Locally coalgebra-Galois extensions

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Abstract

The paper introduces the notion of a *locally coalgebra-Galois extension* and, as its special case, a *locally cleft extension*, generalising concepts from [9]. The necessary and sufficient conditions for a locally coalgebra-Galois extension to be a (global) coalgebra-Galois extension are stated. As an important special case, it is proven, that under not very restrictive conditions the gluing of two locally cleft extensions is a globally coalgebra-Galois extension. As an example, the quantum lens space of positive charge is constructed by gluing of two quantum solid tori.

1 Introduction

Constructing new topological spaces by gluing together several known ones, or studying the properties of a given space by presenting it as a patching of topological spaces of a simpler structure, is the standard method in the classical topology which was frequently adapted to the noncommutative geometry. Examples include the quantum real projective sphere ([13]), and the Podleś sphere (defined in [20], it was proven in [22] and [13] that it is C^*-isomorphic with gluing of two quantum discs), the Matsumoto sphere and the quantum lens space ([16] and [17]).

The basic idea about covering quantum spaces by quantum subsets stems from the observation that an ideal of the algebra of functions on a given quantum space can be interpreted as consisting of those functions, which assume the value zero on some quantum subset. Then the quotient algebra can be viewed as the algebra of functions on this quantum subset.

Suppose that the algebra of functions on some quantum space has a family of ideals, which intersect at zero. Dually it means that the corresponding quantum subsets cover the whole of quantum space.

The covering and gluing of C^*-algebras and the notion of a locally trivial quantum principal bundles in the context of C^*-algebras were introduced in [7]. The purely algebraic theory of covering and gluing of algebras and differential algebras was presented in [8], which was followed by the purely algebraic definition

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of locally trivial quantum principal bundles in [9]. An example of a locally trivial principal bundle was produced in [9] and further elaborated in [12]. The theory resembling locally cleft extensions, based on sheaf theory was independently developed in [19].

In what follows we introduce the concept of *locally coalgebra-Galois extensions* which, without going into technical details, are the algebras and comodules which have covers such that each of the quotient spaces are coalgebra-Galois extensions. Of a particular interest are the conditions which ensure that a locally coalgebra-Galois extension is a (global) coalgebra-Galois extension. Later we concentrate on the special case, in which all the quotient spaces of the cover of a locally Galois extension are cleft, which can be considered as the natural generalisation of locally trivial quantum principal bundles introduced in [9].

2 Basic definitions and notation

We work over the general commutative ring $\mathbb{K}$.

Categories of modules and comodules. Let C and H be coalgebras, and A and B algebras. We denote by ${}_A\mathcal{M}$, $\mathcal{M}_B$, ${}^H\mathcal{M}$, $\mathcal{M}^C$, ${}_A\mathcal{M}_B$, ${}^H\mathcal{M}^C$, ${}_A\mathcal{M}^C$, etc, respectively, the category of left A-modules, right B-modules, left H-comodules, right C-comodules, (A, B)-bimodules, (H, C)-bicomodules, left A-modules and right C-comodules such that the C-coaction commutes with the A-action, etc. Unadorned $\mathcal{M}$ will denote category of $\mathbb{K}$-modules. All algebraic objects considered belong to this category.

Comultiplication, coaction and the Sweedler notation. Suppose C and H are coalgebras, $M \in {}^H\mathcal{M}$ and $N \in \mathcal{M}^C$. We denote the comultiplication by $\Delta : C \to C \otimes C$, the left H-coaction by ${}^H\rho : M \to H \otimes M$, the right C-coaction by $\rho^C : N \to N \otimes C$. Occasionally, to avoid confusion, we indicate the module being coacted on, writing ρ^C_N for $\rho^C : N \to N \otimes C$, and similarly for left coactions. We also use the Sweedler notation: $\Delta(c) = c_{(1)} \otimes c_{(2)}$, ${}^H\rho(m) = m_{(-1)} \otimes m_{(0)}$, $\rho^C(n) = n_{(0)} \otimes n_{(1)}$ for all $c \in C$, $m \in M$, $n \in N$, and the summation is implicitly understood.

Antipodes, counits and units. Let C be a coalgebra. We denote by $\varepsilon^C : C \to \mathbb{K}$ the counit of C. If there is no danger of confusion, we write ε for ε^C. If C is a Hopf algebra, then the antipode of C is denoted by $S : C \to C$. If A is an algebra, in most cases we use the symbol 1_A or simply 1 for the unit of A. Occasionally, we may need to write the unit explicitly as a map $\eta : \mathbb{K} \to A$, $k \mapsto k1_A$. Whenever it does not cause any ambiguity, we identify in notation the ground ring $\mathbb{K}$ with the subalgebra $\mathbb{K}1_A$ of A, i.e., depending on context, for any $k \in \mathbb{K}$, k may mean also $k1_A$ for some algebra A.

Entwining structures Let C be a coalgebra and A an algebra. Entwining structures were introduced in [5] as a very general way of linking algebra structure on

A and coalgebra structure on C. A good introduction to entwining strtuctures can be found in [6]. Throughout this paper by entwining map we mean a right-right version $\psi : C \otimes A \to A \otimes C$, the corresponding entwining structure is denoted $(A, C)_\psi$.

We use the following summation notation for an entwining map ψ:

$$\psi(c \otimes a) = a_\alpha \otimes c^\alpha, \text{ for all } c \in C,\ a \in A,$$

where small Greek letters are used for implicit summation indices.

With this summation notation, the bow-tie diagram condition (cf. [6]) can be explicitly written as

$$1_\alpha \otimes c^\alpha = 1 \otimes c, \tag{1a}$$

$$a_\alpha \varepsilon^C(c^\alpha) = a\varepsilon^C(c), \tag{1b}$$

$$(aa')_\alpha \otimes c^\alpha = a_\alpha a'_\beta \otimes c^{\alpha\beta}, \tag{1c}$$

$$a_\alpha \otimes c^\alpha{}_{(1)} \otimes c^\alpha{}_{(2)} = a_{\beta\alpha} \otimes c_{(1)}{}^\alpha \otimes c_{(2)}{}^\beta, \tag{1d}$$

for all $a, a' \in A$, $c \in C$.

Let $(A, C)_\psi$ be an entwining structure, and let P be an algebra and an $(A, C)_\psi$-entwined module. An algebra extension $B \subseteq P$ is called an $(A, C)_\psi$-*extension* if and only if $B = P^{\mathrm{co}C}$. Of particular interests are $(A, C)_\psi$-extensions $B \subseteq A$. Such extensions are denoted by $A(B, C, \psi)$. In this case, if there exists a grouplike element $e \in C$ such that, for all $a \in A$, $\rho^C(a) = \psi(e \otimes a)$ then $A_e(B, C, \psi)$ is called an e-copointed $(A, C)_\psi$-extension.

The canonical map and quantum principal bundles Suppose that C is a coalgebra and P is an algebra and a right C-comodule. Let $B = P^{\mathrm{co}C}$ be a subalgebra of coinvariants of right C-coaction. To set the notation we recall the definition of the canonical map,

$$\mathrm{can}_P^C : P \otimes_B P \to P \otimes C,\ \ p \otimes_B p' \mapsto pp'_{(0)} \otimes p'_{(1)}.$$

If the canonical map is a bijection then $P(B)^C$ is called a C-coalgebra Galois extension of B. If, in addition C is a Hopf algebra and P is a C-comodule algebra, then $P(B)^C$ is called a C-Hopf Galois extension. We recall the definition of the translation map,

$$\tau_P^C : C \to P \otimes_B P,\ \ c \mapsto (\mathrm{can}_P^C)^{-1}(1 \otimes c),$$

for all $c \in C$. We use an explicit 'Sweedler like' notation for the translation map,

$$\tau_P^C(c) = c^{[1]} \otimes_B c^{[2]}, \text{ for all } c \in C,$$

where an implicit summation is understood. The translation map has a number of useful properties (cf. 34.4 [6]). In particular, for all $c \in C$, $p \in P$,

$$c^{[1]} c^{[2]}{}_{(0)} \otimes c^{[2]}{}_{(1)} = 1_P \otimes c, \tag{2}$$

$$p_{(0)} p_{(1)}{}^{[1]} \otimes_B p_{(1)}{}^{[2]} = 1_P \otimes_B p, \tag{3}$$

Entwinings and C-coalgebra Galois extensions are closely related. If $P(B)^C$ is a C-coalgebra Galois extension, then the map (cf. Theorem 2.7 [4])

$$\psi_{\text{can}} : C \otimes P \to P \otimes C, \;\; c \otimes a \mapsto \text{can}_P^C(\tau_P^C(c)p) \tag{4}$$

is the unique entwining such that P is an entwined module. In particular, if $P(B)^C$ is a C-Hopf Galois extension, then $\psi_{\text{can}}(c \otimes p) = p_{(0)} \otimes cp_{(1)}$.

Multiplication of subsets Suppose that A is an algebra, $M \in {}_A\mathcal{M}$ and $\rho_A : A \otimes M \to M$ denotes the left A-action. Let $B \subseteq A$, $N \subseteq M$ be subsets. Unless otherwise stated, in what follows, we denote $BN \equiv \rho_A(B \otimes N)$. We use similar convention for right modules.

2.1 Cleft extensions

An interesting class of coalgebra-Galois extensions is provided by cleft extensions.

Definition 2.1. Let C be a coalgebra, P be an algebra and a right C-comodule. A convolution invertible and a right C-colinear map $\gamma : C \to P$ is called a *cleaving map*. A C-coalgebra Galois extension $P(B)^C$ such that there exists a cleaving map $\gamma : C \to P$ is called a *cleft coalgebra Galois extension* and is denoted by $P(B)^C_\gamma$. Similarly a *cleft* $(P,C)_\psi$*-extension* $P_\gamma(B,C,\psi)$ is a $(P,C)_\psi$-extension with a cleaving map γ.

Observe that if $P(B)^C$ is cleft, then the inverse of the canonical map has the form

$$(\text{can}_P^C)^{-1}(p \otimes c) = p\gamma^{-1}(c_{(1)}) \otimes_B \gamma(c_{(2)}), \text{ for all } c \in C, p \in P, \tag{5}$$

where γ^{-1} means the convolution inverse.

Proposition 2.2. *(Proposition 2.3 [3].) Let C be a coalgebra, P be a right comodule and $B = P^{coC}$. If there exists a cleaving map $\gamma : C \to P$, then the following are equivalent:*

1. *$P(B)^C$ is a C-coalgebra Galois extension.*
2. *There exists an entwining ψ such that $P(B,C,\psi)$ is a $(P,C)_\psi$-extension.*
3. *For all $p \in P$, $p_{(0)}\gamma^{-1}(p_{(1)}) \in B$.*

If any of the above conditions hold, then $P \simeq B \otimes C$ in ${}_B\mathcal{M}^C$, where the isomorphism $\theta_\gamma : P \to B \otimes C$ and its inverse $\theta_\gamma^{-1} : B \otimes C \to P$ are given explicitly by

$$\begin{aligned} \theta_\gamma(p) &= p_{(0)}\gamma^{-1}(p_{(1)}) \otimes p_{(2)}, \\ \theta_\gamma^{-1}(b \otimes c) &= b\gamma(c). \end{aligned} \tag{6}$$

Lemma 2.3. *Suppose that $P(B)^C_\gamma$ is a cleft C-coalgebra Galois extension, and that $\gamma : C \to P$ is a cleaving map on P. The map*

$$\gamma' = m \circ (\Gamma \otimes \gamma) \circ \Delta, \tag{7}$$

where $\Gamma : C \to B$ is a convolution invertible map called a gauge transformation, *is also a cleaving map on P, and any other cleaving map on P has this form.*

3 Covering of modules and algebras

In this and the next section we recall basic definitions and theorems from [8]. Note that the covering and gluing of modules was actually introduced in [10].

In what follows, all algebraic objects, unless specified otherwise, are $\mathbb{K}$-modules, where $\mathbb{K}$ is a unital commutative ring such that $\mathbb{K} \ni 2 \neq 0$, $\mathbb{K} \ni 3 \neq 0$, and any $\mathbb{K}$-module M considered is such that $2M = M$, $3M = M$.

Definition 3.1. Let A, B be algebras and let C be a coalgebra. Suppose that M is an (A, B)-bimodule (resp. a right C-comodule, an algebra, an algebra and a right C-comodule, etc.) Let I be a finite index set, and let $(J_i)_{i\in I}$ be a family of sub-bimodules (resp. C-sub-comodules, ideals, ideals which are also right C-sub-comodules, etc.) of M, such that

$$\bigcap_{i\in I} J_i = \{0\}. \tag{8}$$

Then the family $(J_i)_{i\in I}$ is called a *cover* or a *covering* of M.

In what follows we will only consider finite covers, i.e., in the statement '$(J_i)_{i\in I}$ is a covering' it should be implicitly understood that the index set I is finite.

Observe that the quotient modules

$$M_i = M/J_i, \quad M_{ij} = M/(J_i + J_j), \quad M_{ijk} = M/(J_i + J_j + J_k), \quad \ldots, \tag{9}$$

are (A, B)-bimodules (resp. C-comodules, algebras, algebras and right C-comodules, etc.), and hence, for all $i, j, k \in I$, the canonical surjections

$$\begin{gathered}\pi_i : M \to M_i,\ m \mapsto m + J_i; \quad \pi_{ij} : M \to M_{ij},\ m \mapsto m + J_i + J_j;\\ \pi_{ijk} : M \to M_{ijk},\ m \mapsto m + J_i + J_j + J_k; \quad \ldots;\\ \pi^i_j : M_i \to M_{ij},\ m + J_i \mapsto m + J_i + J_j;\\ \pi^i_{jk} : M_i \to M_{ijk},\ m + J_i \mapsto m + J_i + J_j + J_k; \quad \ldots\end{gathered} \tag{10}$$

are morphisms in the respective categories. Note that, by our assumption at the beginning of this section, about the ground ring $\mathbb{K}$, for all $i, j, k \in I$, $M_{ii} = M_i = M_{iii}$, $M_{iij} = M_{ij}$, $M_{ij} = M_{ji}$, etc., and also

$$\pi_{ii} = \pi_i,\ \pi_{ij} = \pi_{ji},\ \pi^i_i = M_i,\ \pi^i_j \circ \pi_i = \pi_{ij},\ \pi^i_{jk} \circ \pi_i = \pi_{ijk},\ \text{etc.}, \tag{11}$$

A module

$$M^{\mathbf{c}} = \{(m_i)_{i\in I} \in \bigoplus_{i\in I} M_i \mid \forall_{i,j\in I}\ \pi^i_j(m_i) = \pi^j_i(m_j)\} \tag{12}$$

is called a *covering completion* of M. Observe that $M^{\mathbf{c}} = \ker \Psi_M$, where

$$\Psi_M : \bigoplus_{i\in I} M_i \to \bigoplus_{i,j\in I} M_{ij}, \quad (m_i)_{i\in I} \mapsto (\pi^i_j(m_i) - \pi^j_i(m_j))_{i,j\in I}. \tag{13}$$

The map

$$\kappa_M : M \to M^{\mathbf{c}}, \;\; m \mapsto (\pi_i(m))_{i\in I} \tag{14}$$

is clearly injective (as $\ker \kappa_M = \bigcap_{i\in I} \ker \pi_i = \bigcap_{i\in I} J_i = \{0\}$). If κ_M is also surjective, then the cover $(J_i)_{i\in I}$ of M is called *a complete cover*.

Note that the definitions of the module $M^{\mathbf{c}}$ and the map κ_M make sense even if the family $(J_i)_{i\in I}$ is not a cover. Accordingly, in what follows, we shall make use of the term covering completion $M^{\mathbf{c}}$ and the map κ_M also when $\bigcap_{i\in I} J_i \neq \{0\}$. In fact, κ_M is injective if and only if $(J_i)_{i\in I}$ is a cover.

If M is an algebra with unit (and J_i, $i \in I$, are ideals), then $M^{\mathbf{c}}$ is an algebra, with unit $(1_{M_i})_{i\in I}$ and

$$(m_i)_{i\in I} \cdot (n_j)_{j\in I} = (m_i n_i)_{i\in I}, \;\; \text{for all } (m_i)_{i\in I}, (n_j)_{j\in I} \in M^{\mathbf{c}}. \tag{15}$$

The map κ_M is then an algebra morphism. Similarly, if C is a coalgebra, flat as a $\mathbb{K}$-module, and M is a C-comodule (and $(J_i)_{i\in I}$ is a family of sub-comodules), then $M^{\mathbf{c}}$ is naturally a C-comodule with the coaction

$$\rho^C : M^{\mathbf{c}} \to M^{\mathbf{c}} \otimes C, \;\; (m_i)_{i\in I} \mapsto (m_{i(0)} \otimes m_{i(1)})_{i\in I}. \tag{16}$$

With respect to this coaction κ_M is a right C-colinear map.

The following two propositions give criterions for a cover to be a complete one.

Lemma 3.2. *(Proposition 1, [8].) Let M be a $\mathbb{K}$-module, and let $J_1, J_2 \subseteq M$ be $\mathbb{K}$-submodules. Then the map $\kappa_M : M \to M^{\mathbf{c}}$ defined by (14) is surjective. In particular, any covering by two subspaces is complete.*

Lemma 3.3. *(Proposition 3, [8].) Let M be a $\mathbb{K}$-module and let $(J_i)_{i\in I}$ be a covering of M. Assume that the index set is $I = \{1, 2, \ldots, n\}$ and that, for all $k \in I$, the submodules M_i satisfy*

$$\bigcap_{i\in\{1,2,\ldots,k-1\}} (J_i + J_k) = \left(\bigcap_{i\in\{1,2,\ldots,k-1\}} J_i \right) + J_k. \tag{17}$$

Then the covering $(J_i)_{i\in I}$ is complete.

The condition (17) is not necessary. Note however that the closed ideals of a C^*-algebra form a net with respect to intersection and addition, which is stronger condition than (17). Moreover, a similar but weaker than (17) condition is a necessary condition.

Lemma 3.4. *(Proposition 4, [8].) Let M be a $\mathbb{K}$-module, and let $(J_i)_{i\in I}$ be a complete covering of M. Then, for all $k \in I$,*

$$\bigcap_{i\neq k} (J_i + J_k) = \left(\bigcap_{i\neq k} J_i \right) + J_k. \tag{18}$$

4 Gluing of modules and algebras

Let M_i, M_{ij}, $i, j \in I$, be a finite family of modules, and let $\pi^i_j : M_i \to M_{ij}$ be a family of surjective homomorphisms such that $M_{ii} = M_i$, $M_{ij} = M_{ji}$ and $\pi^i_i = M_i$, for all $i, j \in I$. Then the module

$$\bigoplus_{\pi^i_j} M_i = \left\{ (m_i)_{i\in I} \in \bigoplus_{i\in I} M_i \mid \forall_{i,j\in I} \pi^i_j(m_i) = \pi^j_i(m_j) \right\} \tag{19}$$

is called a *gluing of the modules* M_i *with respect to* π^i_j (Definition 3 [8]).

Similarly as in the case of covering completions, if the modules M_i, M_{ij}, $i, j \in I$ are (unital) algebras and maps π^i_j are (unital) algebra maps, then gluing $\bigoplus_{\pi^i_j} M_i$ is an algebra. If C is a coalgebra flat as a $\mathbb{K}$-module, and modules M_i, M_{ij} are right C-comodules and the maps π^i_j are right C-colinear, then $\bigoplus_{\pi^i_j} M_i$ is naturally a right C-comodule.

Proposition 4.1. *(Proposition 8 [8].) Suppose that* $M = \bigoplus_{\pi^i_j} M_i$*. For all* $i \in I$*, let*

$$p_i : \bigoplus_{\pi^k_l} M_k \to M_i,\ (m_j)_{j\in I} \mapsto m_i. \tag{20}$$

Then $(\ker(p_i))_{i\in I}$ *is a complete covering of* M*.*

The maps p_i, $i \in I$, defined above are not in general surjective. The reason, given in [8], is that our definition of gluing (19) does not exclude self-gluing. The following proposition gives sufficient condition for surjectivity of maps p_i, $i \in I$.

Proposition 4.2. *(Proposition 9 [8].) Let* $M = \bigoplus_{\pi^i_j} M_i$*. Assume that the epimorphisms* $\pi^i_j : M_i \to M_{ij}$ *have the following properties.*

$$\text{For all } i, j, k \in I,\ \ \pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k). \tag{21}$$

Define isomorphisms

$$\begin{aligned} \theta^{ij}_k &: M_i/(\ker \pi^i_j + \ker \pi^i_k) \to M_{ij}/\pi^i_j(\ker \pi^i_k), \\ m_i &+ \ker \pi^i_j + \ker \pi^i_k \mapsto \pi^i_j(m_i) + \pi^i_j(\ker \pi^i_k). \end{aligned} \tag{22}$$

Then assume that the isomorphisms

$$\phi^k_{ij} = (\theta^{ij}_k)^{-1} \circ \theta^{ji}_k : M_j/(\ker \pi^j_i + \ker \pi^j_k) \to M_i/(\ker \pi^i_j + \ker \pi^i_k) \tag{23}$$

satisfy

$$\phi^j_{ik} = \phi^k_{ij} \circ \phi^i_{jk},\ \text{for all } i, j, k \in I. \tag{24}$$

Let $I = \{1, 2, \ldots, n\}$. If $n > 3$, assume that, for all $1 \leq k < n$ and $1 \leq i < k$,

$$\bigcap_{1\leq j\leq i} (\ker \pi_j^{k+1} + \ker \pi_{i+1}^{k+1}) = \left(\bigcap_{1\leq j\leq i} \ker \pi_j^{k+1} \right) + \ker \pi_{i+1}^{k+1}. \tag{25}$$

Then, for all $i \in I$, the maps p_i (20) are surjective.

Remark. Note that, for all $i, j, k \in I$, $m_j \in M_j$, $\phi_{ij}^k(m_j + \ker(\pi_i^j) + \ker(\pi_k^j)) = m_i + \ker(\pi_j^i) + \ker(\pi_k^i)$, where m_i is any element of M_i such that $\pi_j^i(m_i) = \pi_i^j(m_j)$.

5 Locally C-coalgebra Galois extensions

In what follows we shall frequently use the following two simple observations.

Lemma 5.1. *Suppose that A and B are algebras, and that $\pi : A \to B$ is a surjective algebra morphism. Take any $N \in {}_B\mathcal{M}_B$. Clearly $N \in {}_A\mathcal{M}_A$ with left and right A-actions defined by $a \cdot n \cdot a' = \pi(a)n\pi(a')$, for all $a, a' \in A$, $n \in N$. Then we can identify $N \otimes_B N$ with $N \otimes_A N$.*

Lemma 5.2. *Suppose that $P(B)^C$ is a C-coalgebra Galois extension. Let A be an algebra and a right C-comodule, and suppose that $\pi : P \to A$ is an algebra and a right C-comodule morphism. If $\pi(B) = A^{coC}$, then $A(\pi(B))^C$ is a C-coalgebra Galois extension.*

Proof. It is clear that the map

$$\tau_A^C = (\pi \otimes_B \pi) \circ \tau_P^C : C \to A \otimes_B A \simeq A \otimes_{\pi(B)} A \tag{26}$$

is the translation map on A, where τ_P^C is the translation map on P, and we used the identification of $A \otimes_B A$ with $A \otimes_{\pi(B)} A$ (Lemma 5.1). $\square$

Observe that it is not true in general that for an arbitrary coalgebra C and algebras and right C-comodules P and A, such that there exists an algebra surjection $\pi : P \to A$, we have $A^{\mathrm{co}C} = \pi(P^{\mathrm{co}C})$. As an example take C being a commutative Hopf algebra generated by a single primitive element x, i.e., $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\varepsilon(x) = 0$. Let P be a free commutative algebra generated by two elements b and a, and let us define a right C-coaction $\rho^C : P \to P \otimes C$ as an algebra map defined by an algebra extension of the relations

$$\rho^C(b) = b \otimes 1, \;\; \rho^C(a) = a \otimes 1 + b \otimes x. \tag{27}$$

It is clear that $P^{\mathrm{co}C}$ is a subalgebra of P generated by the element b. Let $A = P/(Pb)$, i.e., A is a free commutative algebra generated by $a + Pb$, and let $\pi : P \to A$ be the canonical surjection on the quotient space. The map π is clearly right C-colinear, and moreover, C acts trivially on A, i.e, $A^{\mathrm{co}C} = A$. However, $\pi(P^{\mathrm{co}C}) = \mathbb{K}$.

Unfortunately P is not a C-Hopf Galois extension, and we were unable either to prove that $A^{\mathrm{co}C} = \pi(P^{\mathrm{co}C})$ when P is a C-coalgebra Galois extension, nor to produce a counterexample. The following lemma, however, shows that, under a not very restrictive condition, $A^{\mathrm{co}C} = \pi(P^{\mathrm{co}C})$ when P is a cleft extension.

Lemma 5.3. *Suppose that $P(B)^C$ is a cleft C-coalgebra Galois extension, and that $\pi : P \to A$ is a surjective algebra and right C-comodule morphism. Let $\gamma_P : C \to P$ be a cleaving map in P. If $\pi(1_{(0)}\gamma_P^{-1}(1_{(1)}))$ has a right inverse in $\pi(B)$, then $A(\pi(B))^C$ is a cleft C-coalgebra Galois extension.*

Proof. The map $\gamma_A = \pi \circ \gamma_P : C \to A$ is right C-colinear as the composition of two C-colinear maps, and it is convolution invertible, with the convolution inverse given by $\gamma_A^{-1} = \pi \circ \gamma_P^{-1}$. Moreover, since π is surjective, for all $a \in A$, there exists $p \in P$ such that $a = \pi(p)$, and then

$$a_{(0)}\gamma_A^{-1}(a_{(1)}) = \pi(p_{(0)}\gamma_P^{-1}(p_{(1)})) \in \pi(B) \subseteq A^{\mathrm{co}C}.$$

Therefore, by Proposition 2.2, it remains to prove that $A^{\mathrm{co}C} = \pi(B)$. Consider the map

$$B \otimes C \xrightarrow{b\otimes c \mapsto b\gamma_P(c)} P \xrightarrow{\pi} A \tag{28}$$

which is surjective by Proposition 2.2. Therefore, in particular, for all $s \in A^{coC}$, there exists $\sum_i b_i \otimes c_i \in B \otimes C$, such that $s = \pi(\sum_i b_i\gamma_P(c_i)) = \sum_i \pi(b_i)\gamma_A(c_i)$. The C-coinvariants of A are characterised by the property $\rho^C(s) = s\rho^C(1)$, therefore,

$$\sum_i \pi(b_i)\gamma_A(c_i)1_{(0)} \otimes 1_{(1)} = \sum_i \pi(b_i)\gamma_A(c_{i(1)}) \otimes c_{i(2)}.$$

Applying $m \circ (P \otimes \gamma_A^{-1})$ to both sides of the above equation yields

$$\sum_i \pi(b_i)\gamma_A(c_i)1_{(0)}\gamma_A^{-1}(1_{(1)}) = \sum_i \pi(b_i)\varepsilon(c_i),$$

hence, if $1_{(0)}\gamma_A^{-1}(1_{(1)})$ has a right inverse $R \in \pi(B)$, then

$$s = \sum_i \varepsilon(c_i)\pi(b_i)R \in \pi(B),$$

which ends the proof. □

Note that if $P(B)^C_{e,\gamma_P}$ is an $e-$*copointed* cleft C-coalgebra Galois extension, then $1_{(0)}\gamma_P^{-1}(1_{(1)}) = \gamma_P^{-1}(e)$, which is invertible in B with $(\gamma_P^{-1}(e))^{-1} = \gamma_P(e) \in B$.

Definition 5.4. A pair $(P(B)^C, (J_i)_{i\in I})$ is called a *locally C-coalgebra Galois extension* if the following conditions are satisfied.

1. P is an algebra and a right C-comodule and $B = P^{\mathrm{co}C}$.

2. The family $(J_i)_{i\in I}$ of ideals and right C-subcomodules of P is a complete cover of the algebra P.

3. For all $i \in I$, $\pi_i(B) = P_i^{\mathrm{co}C}$, and $P_i(\pi_i(B))^C$ is a C-coalgebra Galois extension.

4. For all $i, j \in I$, $\pi_{ij}(B) = P_{ij}^{\mathrm{co}C}$.

Note that while $(B \cap J_i)_{i \in I}$ is a cover of B it does not need to be a complete cover. Indeed, in general $B \cap J_i + B \cap J_j \neq B \cap (J_i + J_j)$, therefore $\bar{\pi}^i_j \neq \pi^i_j\Big|_{B/(B\cap J_i)}$, where $\bar{\pi}^i_j : B/(B \cap J_i) \to B/(B \cap J_i + B \cap J_j), b + B \cap J_i \mapsto b + B \cap J_i + B \cap J_j$.

The following lemma is very technical and apparently obvious. However, we shall make use of it several times in critical places and we want to state it explicitly.

Lemma 5.5. *Let I, J be index sets and suppose that C, M_i, $i \in I$, N_j, $j \in J$, are $\mathbb{K}$-modules. There is a well known canonical identification*

$$\vartheta_M : (\bigoplus_{i\in I} M_i) \otimes C \to \bigoplus_{i\in I}(M_i \otimes C),\ (m_i)_{i\in I} \otimes c \mapsto (m_i \otimes c)_{i\in I},$$

and similarly $\vartheta_N : (\bigoplus_{j\in J} N_j) \otimes C \simeq \bigoplus_{j\in J}(N_j \otimes C)$. Let $F^i_j : M_i \to N_j$, $i \in I$, $j \in J$ be a family of $\mathbb{K}$-linear morphisms. Define maps

$$F : \bigoplus_{i\in I} M_i \to \bigoplus_{j\in J} N_j,\ \ (m_i)_{i\in I} \mapsto (\sum_{i\in I} F^i_j(m_i))_{j\in J}, \tag{29}$$

$$G : \bigoplus_{i\in I}(M_i \otimes C) \to \bigoplus_{j\in J}(N_j \otimes C),\ (m_i \otimes c_i)_{i\in I} \mapsto (\sum_i F^i_j(m_i) \otimes c_i)_{j\in J}. \tag{30}$$

Then $G \circ \vartheta_M = \vartheta_N \circ (F \otimes C)$.

Lemma 5.6. *Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a locally C-coalgebra Galois extension. For all $i \in I$, denote by $\tau_i : C \to P_i \otimes_B P_i$ the translation map in P_i. Then, for all $i, j \in I$,*

$$(\pi^i_j \otimes_B \pi^i_j) \circ \tau_i = (\pi^j_i \otimes_B \pi^j_i) \circ \tau_j. \tag{31}$$

Proof. By Lemma 5.2, both sides of (31) are translation maps in P_{ij}. But the translation map, if it exists, is unique, hence the equality. □

We use an indexed summation notation for the translation map. For all $i \in I$ and $c \in C$,

$$\tau_i(c) = c^{[1]_i} \otimes_B c^{[2]_i}, \tag{32}$$

implicit summation (not over i though!) is understood.

Proposition 5.7. *Let $(P(B)^C, (J_i)_{i\in I})$ be a locally C-coalgebra Galois extension and suppose that C is flat as a $\mathbb{K}$-module. Then P is a $(P, C)_\psi$-entwined module with*

$$\psi : C \otimes P \to P \otimes C,\ \ c \otimes p \mapsto (\kappa_P^{-1} \otimes C)((\psi_i(c \otimes \pi_i(p)))_{i\in I}), \tag{33}$$

where, for all $i \in I$, ψ_i is the canonical entwining on P_i (4).

Proof. First we prove that the map ψ is well defined. Using (31) we show that, for all $i, j \in I$, $c \in C$, $p \in P$,

$$(\pi^i_j \otimes C) \circ \psi_i(c \otimes \pi_i(p)) = (\pi^j_i \otimes C) \circ \psi_j(c \otimes \pi_j(p)). \tag{34}$$

Indeed,

$$
\begin{aligned}
(\pi^i_j \otimes C) \circ \psi_i(c \otimes \pi_i(p)) &= (\pi^i_j \otimes C)(c^{[1]_i}\rho^C(c^{[2]_i}\pi_i(p))) \\
&= \pi^i_j(c^{[1]_i})\rho^C(\pi^i_j(c^{[2]_i})\pi^i_j(\pi_i(p))) = \pi^j_i(c^{[1]_j})\rho^C(\pi^j_i(c^{[2]_j})\pi^j_i(\pi_j(p))) \\
&= (\pi^j_i \otimes C)(c^{[1]_j}\rho^C(c^{[2]_j}\pi_j(p))) = (\pi^j_i \otimes C) \circ \psi_j(c \otimes \pi_j(p)).
\end{aligned}
$$

Define the map

$$
\bar{\psi} : C \otimes P \to \bigoplus_{i \in I}(P_i \otimes C), \ \ c \otimes p \mapsto (\psi_i(c \otimes \pi_i(p)))_{i \in I}.
$$

By (34) and Lemma 5.5, $\mathrm{Im}(\bar{\psi}) \subseteq \ker(\Psi_P \otimes C) = \ker(\Psi_P) \otimes C = P^{\mathbf{c}} \otimes C$, where Ψ_P is as in (13), and we used the flatness of C and the definition of $P^{\mathbf{c}} = \ker(\Psi_P)$. Hence the map $\psi = (\kappa_P^{-1} \otimes C) \circ \bar{\psi}$ is well defined.

In order to distinguish between different entwining maps we use indexed summation notation $\psi_i(c \otimes \pi_i(p)) = \pi_i(p)_{\alpha_i} \otimes c^{\alpha_i}$, for all $i \in I$, $p \in P$, $c \in C$, together with the usual notation $\psi(c \otimes p) = p_\alpha \otimes c^\alpha$, an implicit summation understood. We need to check whether ψ satisfies conditions (1). Indeed, for any $c \in C$,

$$
\psi(c \otimes 1_P) = (\kappa_P^{-1} \otimes C)((\psi_i(c \otimes 1_{P_i}))_{i \in I})(\kappa_P^{-1} \otimes C)((1_{P_i})_{i \in I} \otimes c) = 1_P \otimes c,
$$

where we used that κ_P (hence κ_P^{-1}) is a unital map. Similarly, for all $c \in C$, $p \in P$,

$$
\begin{aligned}
(P \otimes \varepsilon) \circ \psi(c \otimes p) &= (\kappa_P^{-1} \otimes \varepsilon)((\psi_i(c \otimes \pi_i(p)))_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C)(((P_i \otimes \varepsilon) \circ \psi_i(c \otimes \pi_i(p)))_{i \in I}) = \kappa_P^{-1}((\pi_i(p))_{i \in I}\varepsilon(c)) = p\varepsilon(c).
\end{aligned}
$$

Observe that, for all $i \in I$,

$$
(\pi_i \otimes C) \circ \psi = \psi_i \circ (C \otimes \pi_i). \tag{35}
$$

Explicitly, for all $c \in C$, $p \in P$, $i \in I$, $\pi_i(p_\alpha) \otimes c^\alpha = \pi_i(p)_{\alpha_i} \otimes c^{\alpha_i}$. Hence, for all $c \in C$, $p, p' \in P$,

$$
\begin{aligned}
(pp')_\alpha \otimes c^\alpha &= (\kappa_P^{-1} \otimes C)((\pi_i(pp')_{\alpha_i} \otimes c^{\alpha_i})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C)((\pi_i(p)_{\alpha_i}\pi_i(p')_{\beta_i} \otimes c^{\alpha_i\beta_i})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C)((\pi_i(p_\alpha)\pi_i(p'_\beta) \otimes c^{\alpha\beta})_{i \in I}) = p_\alpha p'_\beta \otimes c^{\alpha\beta}.
\end{aligned}
$$

Similarly, for all $p \in P$, $c \in C$,

$$
\begin{aligned}
p_\alpha \otimes c^\alpha{}_{(1)} \otimes c^\alpha{}_{(2)} &= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p_\alpha) \otimes c^\alpha{}_{(1)} \otimes c^\alpha{}_{(2)})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p)_{\alpha_i} \otimes c^{\alpha_i}{}_{(1)} \otimes c^{\alpha_i}{}_{(2)})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p)_{\alpha_i\beta_i} \otimes c_{(1)}{}^{\beta_i} \otimes c_{(2)}{}^{\alpha_i})_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C \otimes C)((\pi_i(p_{\alpha\beta}) \otimes c_{(1)}{}^{\beta} \otimes c_{(2)}{}^{\alpha})_{i \in I}) = p_{\alpha\beta} \otimes c_{(1)}{}^{\beta} \otimes c_{(2)}{}^{\alpha}.
\end{aligned}
$$

It remains to prove that P is an entwined module. Indeed, using the coaction (16), for all $p, p' \in P$,

$$\begin{aligned}
&\rho^C(pp') = \rho^C(\kappa_P^{-1} \circ \kappa_P(pp')) = (\kappa_P^{-1} \otimes C) \circ \rho^C \circ \kappa_P(pp') \\
&= (\kappa_P^{-1} \otimes C)((\rho^C(\pi_i(p)\pi_i(p')))_{i \in I}) = (\kappa_P^{-1} \otimes C)((\pi_i(p)_{(0)}\psi_i(\pi_i(p)_{(1)} \otimes \pi_i(p')))_{i \in I}) \\
&= (\kappa_P^{-1} \otimes C)((\pi_i(p_{(0)})\psi_i(p_{(1)} \otimes \pi_i(p')))_{i \in I}) = p_{(0)}(\kappa_P^{-1} \otimes C)((\psi_i(p_{(1)} \otimes \pi_i(p')))_{i \in I}) \\
&\hspace{20em} = p_{(0)}\psi(p_{(1)} \otimes p').
\end{aligned}$$

□

Although a locally coalgebra Galois extension is built out of Galois extensions it is not necessarily a (global) coalgebra Galois extension. The next theorem, which is the main result of this section, gives (sufficient and necessary in the case of a flat coalgebra) conditions for when a locally coalgebra Galois extension is a global coalgebra Galois extension.

Theorem 5.8. *Let $(P(B)^C, (J_i)_{i \in I})$ be a locally C-coalgebra Galois extension.*

1. *If $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a complete covering of $P \otimes_B P$ and*
$$\ker(\pi_{ij} \otimes_B \pi_{ij}) = \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j), \tag{36}$$
then $P(B)^C$ is a C-coalgebra Galois extension.

2. *Suppose that the coalgebra C is flat as a $\mathbb{K}$ module. The family $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a cover of $P \otimes_B P$ if and only if can_P^C is injective.*

3. *If $P(B)^C$ is a C-coalgebra Galois extension and C is flat as a $\mathbb{K}$-module, then the condition (36) is satisfied and $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a complete covering of $P \otimes_B P$.*

Proof. Observe that, for all $i, j \in I$, modules $P_i \otimes_B P_i$ and $P_{ij} \otimes_B P_{ij}$, as the images of the surjective maps $\pi_i \otimes_B \pi_i$ and $\pi_{ij} \otimes_B \pi_{ij}$ respectively, can be identified with the respective quotient spaces $P \otimes_B P / \ker(\pi_i \otimes_B \pi_i)$ and $P \otimes_B P / \ker(\pi_{ij} \otimes_B \pi_{ij})$. Under this identification the maps $\pi_i \otimes_B \pi_i$ and $\pi_{ij} \otimes_B \pi_{ij}$ can be viewed as quotient maps.

1). Suppose first that $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a complete cover of $P \otimes_B P$ and that relation (36) is satisfied. The condition (36) means that $P_{ij} \otimes_B P_{ij}$ can be identified with $P \otimes_B P / (\ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j))$. The map $\pi^i_j \otimes_B \pi^i_j$ is surjective and $(\pi^i_j \otimes_B \pi^i_j) \circ (\pi_i \otimes_B \pi_i) = \pi_{ij} \otimes_B \pi_{ij}$. Therefore, the map $\pi^i_j \otimes_B \pi^i_j$ can be viewed as a quotient map

$$\begin{aligned}
\pi^i_j \otimes_B \pi^i_j : P_i \otimes_B P_i &\to P_{ij} \otimes_B P_{ij}, \\
x + \ker(\pi_i \otimes_B \pi_i) &\mapsto x + \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j).
\end{aligned} \tag{37}$$

It follows that the covering completion of $P \otimes_B P$ can be equivalently defined as

$$(P \otimes_B P)^{\mathbf{c}} = \{(x_i)_{i \in I} \in \bigoplus_{i \in I} P_i \otimes_B P_i \mid \forall_{i,j \in I} (\pi^i_j \otimes_B \pi^i_j)(x_i) = (\pi^j_i \otimes_B \pi^j_i)(x_j)\}, \tag{38}$$

and then, by assumption, the map (cf. (14))

$$\kappa_{P\otimes_B P} : P \otimes_B P \mapsto (P \otimes_B P)^{\mathbf{c}}, \;\; x \mapsto ((\pi_i \otimes_B \pi_i)(x))_{i\in I} \tag{39}$$

is bijective. Define a map

$$\tau^{\mathbf{c}} : C \to (P \otimes_B P)^{\mathbf{c}}, \;\; c \mapsto (\tau_i(c))_{i\in I}, \tag{40}$$

where $\tau_i : C \mapsto P_i \otimes_B P_i$ is the translation map on P_i, $i \in I$. Equation (31) ensures that this map has image in $(P \otimes_B P)^{\mathbf{c}}$. We claim that the map

$$(\mathrm{can}_P^C)^{-1} : P \otimes C \to P \otimes_B P, \;\; p \otimes c \mapsto p\kappa^{-1}_{P\otimes_B P} \circ \tau^{\mathbf{c}}(c) \tag{41}$$

is the inverse of the canonical map $\mathrm{can}_P^C : P \otimes_B P \to P \otimes C$ of P. Indeed, denote

$$\bar{\mathrm{can}} : (P \otimes_B P)^{\mathbf{c}} \to \bigoplus_{i\in I} P_i \otimes C, \;\; (p_i \otimes_B q_i)_{i\in I} \mapsto (p_i q_{i(0)} \otimes q_{i(1)})_{i\in I}. \tag{42}$$

It is easy to see that

$$(\kappa_P \otimes C) \circ \mathrm{can}_P^C = \bar{\mathrm{can}} \circ \kappa_{P\otimes_B P}, \tag{43}$$

and therefore, for all $p \in P$ and $c \in C$,

$$\begin{aligned}\mathrm{can}_P^C \circ (\mathrm{can}_P^C)^{-1}(p \otimes c) &= (\kappa_P^{-1} \otimes C) \circ \bar{\mathrm{can}} \circ \kappa_{P\otimes_B P}(p\kappa^{-1}_{P\otimes_B P} \circ \tau^{\mathbf{c}}(c)) \\ &= (\kappa_P^{-1} \otimes C) \circ \bar{\mathrm{can}}((\pi_i(p)\tau_i(c))_{i\in I}) = (\kappa_P^{-1} \otimes C)((\pi_i(p) \otimes c)_{i\in I}) = p \otimes c.\end{aligned}$$

Similarly, for all $p, q \in P$,

$$\begin{aligned}(\mathrm{can}_P^C)^{-1} \circ \mathrm{can}_P^C(p \otimes_B q) &= (\mathrm{can}_P^C)^{-1}(pq_{(0)} \otimes q_{(1)}) = pq_{(0)}\kappa^{-1}_{P\otimes_B P} \circ \tau^{\mathbf{c}}(q_{(1)}) \\ &= \kappa^{-1}_{P\otimes_B P}((\pi_i(p)\pi_i(q)_{(0)}\tau_i(\pi_i(q)_{(1)}))_{i\in I}) = \kappa^{-1}_{P\otimes_B P}(\pi_i(p) \otimes_B \pi_i(q)) = p \otimes_B q,\end{aligned}$$

where in the fourth equality we used (3), and for the third equality we observed that $p\kappa^{-1}_{P\otimes_B P}((x_i)_{i\in I}) = \kappa^{-1}_{P\otimes_B P}((\pi_i(p)x_i)_{i\in I})$, for all $p \in P$, $(x_i)_{i\in I} \in (P \otimes_B P)^{\mathbf{c}}$.

2.) Suppose that C is flat as a $\mathbb{K}$-module. It is clear that

$$\bar{\mathrm{can}}((P \otimes_B P)^{\mathbf{c}}) \subseteq \ker(\Psi_P \otimes C) = P^{\mathbf{c}} \otimes C,$$

where the last equality follows by the flatness of C and the definition of $P^{\mathbf{c}}$ (38) and Ψ_P (13). Define

$$\mathrm{can}^{\mathbf{c}} : (P \otimes_B P)^{\mathbf{c}} \to P^{\mathbf{c}} \otimes C, \;\; x \mapsto \bar{\mathrm{can}}(x). \tag{44}$$

It is clear that $\mathrm{can}^{\mathbf{c}}$ is invertible with the inverse

$$(\mathrm{can}^{\mathbf{c}})^{-1} : P^{\mathbf{c}} \otimes C \to (P \otimes_B P)^{\mathbf{c}}, \;\; (p_i)_{i\in I} \otimes C \to ((\mathrm{can}_{P_i}^C)^{-1}(p_i \otimes c))_{i\in I}. \tag{45}$$

Using (43), and noticing that $\kappa_P \otimes C$ is bijective, it is easy to see that $\kappa_{P\otimes_B P}$ is injective if and only if can_P^C is. But the injectivity of $\kappa_{P\otimes_B P}$ is equivalent to $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ being a cover.

3. For brevity, we denote $\mathrm{can}_i = \mathrm{can}^C_{P_i}$, $\mathrm{can}_{ij} = \mathrm{can}^C_{P_{ij}}$. By Lemma 5.2, for all $i, j \in I$,

$$\mathrm{can}_i \circ (\pi_i \otimes_B \pi_i) = (\pi_i \otimes C) \circ \mathrm{can}^C_P, \tag{46}$$

$$\mathrm{can}_{ij} \circ (\pi_{ij} \otimes_B \pi_{ij}) = (\pi_{ij} \otimes C) \circ \mathrm{can}^C_P. \tag{47}$$

Hence, as the maps can^C_P, can_i, can_{ij} are bijective and C is flat as a $\mathbb{K}$-module, it follows that

$$\begin{aligned}\ker(\pi_{ij} \otimes_B \pi_{ij}) = (\mathrm{can}^C_P)^{-1}(\ker(\pi_{ij} \otimes C)) &= (\mathrm{can}^C_P)^{-1}(\ker(\pi_{ij}) \otimes C)\\ &= (\mathrm{can}^C_P)^{-1}(\ker(\pi_i) \otimes C + \ker(\pi_j) \otimes C)\\ &= (\mathrm{can}^C_P)^{-1}(\ker(\pi_i \otimes C)) + (\mathrm{can}^C_P)^{-1}(\ker(\pi_j \otimes C))\\ &= \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j).\end{aligned}$$

Furthermore, by (43), as the maps $\mathrm{can}^{\mathbf{c}}$, $\kappa_P \otimes C$ and can^C_P are bijective, we obtain that

$$\kappa_{P \otimes_B P} = (\mathrm{can}^{\mathbf{c}})^{-1} \circ (\kappa_P \otimes C) \circ \mathrm{can}^C_P$$

is invertible, hence the family $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a complete cover of $P \otimes_B P$. $\square$

Lemma 5.9. *Let $(P(B)^C, (J_i)_{i\in I})$ be a locally C-coalgebra Galois extension, and suppose that the ground ring $\mathbb{K}$ is a field. Then the condition (36) is satisfied.*

Proof. For all $i, j, k \in I$, the diagrams

$$\begin{tikzcd} 0 \arrow[r] & PdBP \arrow[r] \arrow[d, "\pi_i\otimes\pi_i|_{PdBP}"] & P \otimes P \arrow[r, "\Theta_P"] \arrow[d, "\pi_i\otimes\pi_i"] & P \otimes_B P \arrow[r] \arrow[d, "\pi_i\otimes_B\pi_i"] & 0 \\ 0 \arrow[r] & P_i dBP_i \arrow[r] & P_i \otimes P_i \arrow[r, "\Theta_{P_i}"] & P_i \otimes_B P_i \arrow[r] & 0 \end{tikzcd} \tag{48}$$

and

$$\begin{tikzcd} 0 \arrow[r] & PdBP \arrow[r] \arrow[d, "\pi_{ij}\otimes\pi_{ij}|_{PdBP}"] & P \otimes P \arrow[r, "\Theta_P"] \arrow[d, "\pi_{ij}\otimes\pi_{ij}"] & P \otimes_B P \arrow[r] \arrow[d, "\pi_{ij}\otimes_B\pi_{ij}"] & 0 \\ 0 \arrow[r] & P_{ij} dBP_{ij} \arrow[r] & P_{ij} \otimes P_{ij} \arrow[r, "\Theta_{P_{ij}}"] & P_{ij} \otimes_B P_{ij} \arrow[r] & 0\,, \end{tikzcd} \tag{49}$$

where $\Theta_M : M \otimes M \to M \otimes_B M$, $M \in {}_B\mathcal{M}_B$ is a natural surjection on the quotient space, and d denotes the universal differential, are clearly commutative and have exact rows.

Since the maps $\pi_i \otimes \pi_i|_{PdBP}$ and $\pi_{ij} \otimes \pi_{ij}|_{PdBP}$ are surjective, the application of the Snake Lemma to the above diagrams yields that the maps

$$\Theta_P|_{\ker(\pi_i\otimes\pi_i)} : \ker(\pi_i \otimes \pi_i) \to \ker(\pi_i \otimes_B \pi_i) \tag{50}$$

and

$$\Theta_P|_{\ker(\pi_{ij}\otimes\pi_{ij})} : \ker(\pi_{ij} \otimes \pi_{ij}) \to \ker(\pi_{ij} \otimes_B \pi_{ij}) \tag{51}$$

are well defined and surjective. Observe that as $\mathbb{K}$ is a field, for all $i, j \in I$,

$$\ker(\pi_i \otimes \pi_i) = \ker(\pi_i) \otimes P + P \otimes \ker(\pi_i),$$

and

$$\begin{aligned}\ker(\pi_{ij} \otimes \pi_{ij}) &= \ker(\pi_{ij}) \otimes P + P \otimes \ker(\pi_{ij}) \\ &= \ker(\pi_i) \otimes P + \ker(\pi_j) \otimes P + P \otimes \ker(\pi_i) + P \otimes \ker(\pi_j) \\ &= \ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j).\end{aligned}$$

Hence, for all $i, j \in I$,

$$\begin{aligned}\ker(\pi_{ij} \otimes_B \pi_{ij}) &= \Theta_P(\ker(\pi_{ij} \otimes \pi_{ij})) \\ &= \Theta_P(\ker(\pi_i \otimes \pi_i)) + \Theta_P(\ker(\pi_j \otimes \pi_j)) = \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j).\end{aligned}$$

□

Therefore, Lemma 5.9 implies that, when working over a field, which is probably the most interesting case from a non-commutative geometry point of view, to verify whether a locally Galois extension is globally Galois, it suffices to check whether the covering $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete covering of $P \otimes_B P$. More precisely we can state:

Corollary 5.10. *Suppose that the ground ring $\mathbb{K}$ is a field and that $(P(B)^C, (J_i)_{i\in I})$ is a locally C-coalgebra Galois extension. Then $P(B)^C$ is a C-coalgebra Galois extension if and only if $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete cover of $P \otimes_B P$.*

In view of Corollary 5.10 it is important to study when a cover is a complete cover.

Lemma 5.11. *Let B be an algebra, and let $(K_i)_{i\in I}$ be a family of ideals of B. Denote the quotient spaces by $B_i = B/K_i$, $B_{ij} = B/(K_i + K_j)$, $i, j \in I$, and by*

$$\pi_i : B \to B_i, \;\; \pi_{ij} : B \to B_{ij}, \;\; \pi^i_j : B_i \to B_{ij}, \quad i, j \in I$$

the canonical surjections. Suppose that $M \in {}_B\mathcal{M}$, $M_i \in {}_{B_i}\mathcal{M}$, $M_{ij} \in {}_{B_{ij}}\mathcal{M}$, $i, j \in I$, is a family of modules such that $M_{ij} = M_{ji}$, for all $i, j \in I$, and that

$$\chi_i : M \to M_i, \;\; \chi_{ij} : M \to M_{ij}, \;\; \chi^i_j : M_i :\to M_{ij}, \quad i, j \in I,$$

is a family of surjective $\mathbb{K}$-linear morphisms with the properties

$$\ker(\chi_{ij}) = \ker(\chi_i) + \ker(\chi_j), \tag{52}$$

$$\chi^i_j \circ \chi_i = \chi_{ij} = \chi^j_i \circ \chi_j, \tag{53}$$

$$\chi_i(bm) = \pi_i(b)\chi_i(m), \;\; \chi_{ij}(bm) = \pi_{ij}(b)\chi_{ij}(m), \tag{54}$$

for all $i, j \in I$, $b \in B$, $m \in M$.

Let $I = \{1, 2, \dots, n\}$, $n > 2$. Suppose that, for all $2 < k \leq n$,

$$\bigcap_{i \in \{1,2,\dots,k-1\}} \chi_k(\ker(\chi_i)) \subseteq \left(\prod_{i=1}^{k-1} \pi_k(K_i) \right) M_k. \tag{55}$$

Then, for all $2 < k \leq n$,

$$\bigcap_{i \in \{1,2,\dots,k-1\}} (\ker(\chi_i) + \ker(\chi_k)) = \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi_i) \right) + \ker(\chi_k). \tag{56}$$

Proof. Note that by (53) and (52), for all $i, j \in I$, $\ker(\chi^i_j) = \chi_i(\ker(\chi_j))$. Therefore, for all $2 < k \leq n$,

$$\begin{aligned}
\bigcap_{i \in \{1,2,\dots,k-1\}} (\ker(\chi_i) + \ker(\chi_k)) &= \bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi_{ik}) = \ker \left(\bigoplus_{i=1}^{k-1} \chi_{ik} \right) \\
&= \ker \left(\left(\bigoplus_{i=1}^{k-1} \chi^k_i \right) \circ \chi_k \right) = (\chi_k)^{-1} \left(\ker \left(\bigoplus_{i=1}^{k-1} \chi^k_i \right) \right) \\
&= (\chi_k)^{-1} \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi^k_i) \right) = (\chi_k)^{-1} \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \chi_k(\ker(\chi_i)) \right) \\
&\subseteq (\chi_k)^{-1} \left(\left(\prod_{i=1}^{k-1} \pi_k(K_i) \right) M_k \right) = \left(\prod_{i=1}^{k-1} K_i \right) M + \ker(\chi_k) \\
&\subseteq \left(\bigcap_{i \in \{1,2,\dots,k-1\}} \ker(\chi_i) \right) + \ker(\chi_k).
\end{aligned}$$

The inclusion relation in the opposite direction is always satisfied. $\square$

Proposition 5.12. *Suppose that the ground ring $\mathbb{K}$ is a field. Let $(P(B)^C, (J_i)_{i \in I})$ be a locally C-coalgebra Galois extension, and let $\pi_i : P \to P_i = P/J_i$, etc., be the surjections on the quotient spaces. Let $K_i = B \cap J_i$, $i \in I$. Suppose that $I = \{1, 2, \dots, n\}$ and, for all $2 < k \leq n$,*

$$\bigcap_{i \in \{1,2,\dots,k-1\}} \pi_k(J_i) \subseteq \left(\prod_{i=1}^{k-1} \pi_k(K_i) \right) P_k. \tag{57}$$

Then if the family $(\ker(\pi_i \otimes_B \pi_i))_{i \in I}$ is a cover of $P \otimes_B P$, then it is a complete cover.

Proof. For all $i, j \in I$, denote $B_i = B/K_i$, $B_{ij} = B/K_{ij}$ and define maps

$$\chi_i = \mathrm{can}_i \circ (\pi_i \otimes_B \pi_i) : P \otimes_B P \to P_i \otimes C, \tag{58}$$
$$\chi_{ij} = \mathrm{can}_{ij} \circ (\pi_{ij} \otimes_B \pi_{ij}) : P \otimes_B P \to P_{ij} \otimes C, \tag{59}$$
$$\chi^i_j = \pi^i_j \otimes C : P_i \otimes C \to P_{ij} \otimes C, \tag{60}$$

which clearly satisfy the conditions (53)-(54). Moreover, as the maps can_i and can_{ij} are bijective, it follows that $\ker(\chi_i) = \ker(\pi_i \otimes_B \pi_i)$ and

$$\ker(\chi_{ij}) = \ker(\pi_{ij} \otimes_B \pi_{ij}) = \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j) = \ker(\chi_i) + \ker(\chi_j),$$

where the second equality follows from Lemma 5.9. Moreover, for all $2 < k \leq n$,

$$\begin{aligned}\bigcap_{i\in\{1,2,\ldots,k-1\}} \chi_k(\ker(\chi_i)) &= \bigcap_{i\in\{1,2,\ldots,k-1\}} \ker(\pi_i^k \otimes C) = \bigcap_{i\in\{1,2,\ldots,k-1\}} \ker(\pi_i^k) \otimes C \\ &= \ker\left(\bigoplus_{i=1}^{k-1} \pi_i^k\right) \otimes C = \left(\bigcap_{i\in\{1,2,\ldots,k-1\}} \pi_k(J_i)\right) \otimes C \subseteq \left(\prod_{i=1}^{k-1} \pi_k(K_i)\right)(P_k \otimes C).\end{aligned}$$

Therefore, by Lemma 5.11, for all $2 < k \leq n$,

$$\begin{aligned}\bigcap_{i\in\{1,2,\ldots,k-1\}} &(\ker(\pi_i \otimes_B \pi_i) + \ker(\pi_k \otimes_B \pi_k)) \\ &= \left(\bigcap_{i\in\{1,2,\ldots,k-1\}} \ker(\pi_i \otimes_B \pi_i)\right) + \ker(\pi_k \otimes_B \pi_k),\end{aligned}$$

and hence, by Lemma 3.3, the family $(\ker(\pi_i \otimes_B \pi_i))_{i\in I}$ is a complete cover of $P \otimes_B P$. ☐

The following lemma is probably well known, but we were unable to find the reference.

Lemma 5.13. *Let M, M', M'' be $\mathbb{K}$-modules and let $K, L \subseteq M$, $K', L' \subseteq M'$, $K'', L'' \subseteq M''$ be submodules. Suppose that*

$$f : K + L \to K' + L', \;\; g : K' + L' \to K'' + L''$$

are $\mathbb{K}$-linear maps such that the sequences

$$0 \longrightarrow K \xrightarrow{f|_K} K' \xrightarrow{g|_{K'}} K'' \longrightarrow 0, \tag{61}$$

$$0 \longrightarrow L \xrightarrow{f|_L} L' \xrightarrow{g|_{L'}} L'' \longrightarrow 0, \tag{62}$$

are well defined and exact. Then the sequence

$$0 \longrightarrow K \cap L \xrightarrow{f|_{K\cap L}} K' \cap L' \xrightarrow{g|_{K'\cap L'}} K'' \cap L'' \longrightarrow 0 \tag{63}$$

is exact if and only if the sequence

$$0 \longrightarrow K + L \xrightarrow{f} K' + L' \xrightarrow{g} K'' + L'' \longrightarrow 0 \tag{64}$$

is exact.

Proof. **(64)⇒(63).** Suppose that the sequence (64) is exact. Clearly $f|_{K\cap L}$ is injective and $g|_{K'\cap L'} \circ f|_{K\cap L} = 0$. Suppose that for some $m' \in K' \cap L'$, $g(m') = 0$. By the exactness of (61) and (62), there exist elements $k \in K$ and $l \in L$, such that $f(k) = m' = f(l)$, i.e., $f(k-l) = 0$. But f is, by the assumption, injective, therefore $k = l \in K \cap L$. Hence the sequence (63) is exact at $K' \cap L'$.

Finally, let $m'' \in K'' \cap L''$. By the exactness of (61) and (62), there exist elements $k \in K'$, $l \in L'$ such that $g(k') = m'' = g(l')$, hence $g(k'-l') = 0$. By the exactness of (64) at $K' + L'$, there exist elements $k \in K$, $l \in L$ such that $f(k+l) = k' - l'$, i.e., $k' - f(k) = l' + f(l) \in K' \cap L'$ and $g(k' - f(k)) = g(k') = m''$, hence $g(K' \cap L') = K'' \cap L''$.

(63)⇒(64). The map g is clearly surjective, as $g(K' + L') = g(K') + g(L') = K'' + L''$ by the exactness of (61) and (62). Similarly, for all $k \in K$, $l \in L$, $g \circ f(k+l) = g \circ f(k) + g \circ f(l) = 0$.

Suppose that, for some $k' \in K'$, $l' \in L'$, $g(k' + l') = 0$, i.e., $g(k') = g(-l') \in K'' \cap L''$. As, by the assumption $g(K' \cap L') = K'' \cap L''$, there exists $m' \in K' \cap L'$ such that $g(m') = g(k') = g(-l')$, i.e., $g(k' - m') = 0$ and $g(l' + m') = 0$. By the exactness of (61) and (62), there exists $k \in K$, $l \in L$, such that $f(k) = k' - m'$ and $f(l) = l' + m'$. Therefore $f(k+l) = k' - m' + l' + m' = k' + l'$, and we have proven that the sequence (64) is exact at $K' + L'$.

Finally, suppose that $f(k+l) = 0$, for some $k \in K$, $l \in L$, i.e, $f(k) = f(-l) \in K' \cap L'$. As $g(f(k)) = 0$, we have by the exactness of (63) at $K' \cap L'$ that there exists $m \in K \cap L$ such that $f(m) = f(k) = f(-l)$, i.e., $f(k-m) = 0$ and $f(l+m) = 0$. However $f|_K$ and $f|_L$ are by assumption injective, hence $k - m = 0$ and $l + m = 0$. Therefore $k + l = k - m + l + m = 0$ and we have proven that f is injective. □

Corollary 5.14. *We keep the notation and assumptions from the above lemma. Suppose that in addition we are given exact sequence of* $\mathbb{K}$*-maps*

$$0 \longrightarrow M \xrightarrow{s} M' \xrightarrow{t} M'' \longrightarrow 0, \tag{65}$$

such that $f = s|_{K+L}$ *and* $g = t|_{K'+L'}$*. Then* $g(K' \cap L') = K'' \cap L''$ *if and only if* $f(K + L) = \ker(g)$*.*

Proof. It is easy to see that, under the assumptions, the sequences (63) and (64) are exact, apart from the conditions $g(K' \cap L') = K'' \cap L''$ and $f(K+L) = \ker(g)$. □

Lemma 5.15. *Suppose that* $\mathbb{K}$ *is a field. Let* $f : M \to N$, $g : M \to N'$ *be* $\mathbb{K}$*-vector space morphisms such that* $\ker(f) \cap \ker(g) = \{0\}$*. Then*

$$\ker(f \otimes f) \cap \ker(g \otimes g) = \ker(f) \otimes \ker(g) + \ker(g) \otimes \ker(f). \tag{66}$$

Proof. As $\mathbb{K}$ is a vector space and $\ker(f) \cap \ker(g) = \{0\}$, we can write

$$M = \bar{M} \oplus \ker(f) \oplus \ker(g),$$

and the assertion of the lemma easily follows. □

Proposition 5.16. *Suppose that $\mathbb{K}$ is a field and that $I = \{1, 2\}$. Let $(P(B)^C, (J_i)_{i\in I})$ be a locally C-coalgebra Galois extension. Suppose that*

$$J_i = (B \cap J_i)P, \quad \text{for all } i \in I. \tag{67}$$

Then $P(B)^C$ is a C-coalgebra Galois extension if and only if

$$(\ker(\pi_1 \otimes \pi_1) + \ker(\pi_2 \otimes \pi_2)) \cap PdBP \\ = \ker(\pi_1 \otimes \pi_1) \cap PdBP + \ker(\pi_2 \otimes \pi_2) \cap PdBP.$$

Proof. By the Snake Lemma, for all $i, j \in I$, the commutative diagrams (48) and (49) with exact rows induce the exact sequences

$$0 \longrightarrow \ker(\pi_i \otimes \pi_i) \cap PdBP \longrightarrow \ker(\pi_i \otimes \pi_i) \longrightarrow \ker(\pi_i \otimes_B \pi_i) \longrightarrow 0 \tag{68}$$

and

$$0 \longrightarrow \ker(\pi_{ij} \otimes \pi_{ij}) \cap PdBP \longrightarrow \ker(\pi_{ij} \otimes \pi_{ij}) \longrightarrow \ker(\pi_{ij} \otimes_B \pi_{ij}) \longrightarrow 0. \tag{69}$$

By Lemma 5.9, the sequence (69) can be written as

$$0 \longrightarrow (\ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j)) \cap PdBP \longrightarrow \ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j) \\ \longrightarrow \ker(\pi_i \otimes_B \pi_i) + \ker(\pi_j \otimes_B \pi_j) \longrightarrow 0.$$

It follows immediately from Corollary 5.14 that

$$\Theta_P(\ker(\pi_i \otimes \pi_i) \cap \ker(\pi_j \otimes \pi_j)) = \ker(\pi_i \otimes_B \pi_i) \cap \ker(\pi_j \otimes_B \pi_j), \tag{70}$$

where $\Theta_P : P \otimes P \to P \otimes_B P$ is the natural surjection on the quotient space (cf. Lemma 5.9), if and only if

$$(\ker(\pi_i \otimes \pi_i) + \ker(\pi_j \otimes \pi_j)) \cap PdBP = \ker(\pi_i \otimes \pi_i) \cap PdBP + \ker(\pi_j \otimes \pi_j) \cap PdBP. \tag{71}$$

Let $I = \{1, 2\}$. Hence $\ker(\pi_1) \cap \ker(\pi_2) = \{0\}$, and so, by Lemma 5.15,

$$\ker(\pi_1 \otimes \pi_1) \cap \ker(\pi_2 \otimes \pi_2) = \ker(\pi_1) \otimes \ker(\pi_2) + \ker(\pi_2) \otimes \ker(\pi_1). \tag{72}$$

Suppose that (67) is satisfied. Then by (72), $\Theta_P(\ker(\pi_i \otimes \pi_i) \cap \ker(\pi_j \otimes \pi_j))\Theta_P(\ker(\pi_1) \otimes \ker(\pi_2) + \ker(\pi_2) \otimes \ker(\pi_1)) = J_1 \otimes_B (J_2 \cap B)P + J_2 \otimes_B (J_1 \cap B)P = 0$ as $J_1J_2, J_2J_1 \subseteq J_1 \cap J_2 = \{0\}$. Therefore, if in addition (70) is satisfied, then $\ker(\pi_1 \otimes_B \pi_1) \cap \ker(\pi_2 \otimes_B \pi_2) = \{0\}$. ☐

6 Locally cleft extensions

Definition 6.1. A locally C-coalgebra Galois extension $(P(B)^C, (J_i)_{i\in I})$ is called a *locally cleft extension* if, for all $i \in I$, the quotient modules P_i are cleft C-coalgebra Galois extensions. It is called a *proper locally cleft extension* if, in addition, for all $i, j \in I$,

$$B \cap (J_i + J_j) = B \cap J_i + B \cap J_j. \tag{73}$$

We adopt the following notation. We denote $P_i = P/J_i$, $P_{ij} = P/(J_i + J_j)$, etc., as before. In addition, we have quotient modules $B_i = B/(B \cap J_i)$, $B_{ij} = B/((J_i \cap B) + (J_j \cap B))$, etc., for all $i, j \in I$. We reserve the use of the Greek letter π with various subscripts and superscripts to surjections onto the quotient modules of B, i.e., $\pi_i : B \to B_i$, $\pi^i_j : B_i \to B_{ij}$, $b + B \cap J_i \mapsto b + B \cap J_i + B \cap J_j$, etc. For quotient maps on P, we use sub- and superscripted letter χ, i.e., $\chi_i : P \to P_i$, etc.

For all $i \in I$, we denote by $\gamma_i : C \to P_i$ a cleaving map on P_i. Moreover, for all $i, j \in I$, we use the notation $\gamma^i_{ij} = \chi^i_j \circ \gamma_i : C \to P_{ij}$.

Note that, for all $i \in I$, $b \in B$, $\chi_i(b) = \pi_i(b)$ if we identify B_i with $B/J_i \subseteq P_i$. Similarly, we can naturally identify $B/(B \cap (J_i + J_j))$ with $B/(J_i + J_j) \subseteq P_{ij}$. In addition, if the relation (73) is satisfied, we can identify B_{ij} with $B/(J_i + J_j)$. Note that the condition (73) is equivalent to

$$\chi^i_j(b) = \pi^i_j(b), \text{ for all } i, j \in I,\ b \in B, \tag{74}$$

where we used the above identifications.

The condition (74) clearly implies that $(B \cap J_i)_{i \in I}$ is a complete cover of B.

Lemma 6.2. *(Cf. Lemma 1 [9].) Let $P(B)^C_\gamma$ be a cleft extension, and let J be an ideal in P such that $\rho^C(J) \subseteq J \otimes C$. Then there exists a left ideal K in B such that $J = K\gamma(C)$. Moreover, if the element $x = 1_{(0)}\gamma^{-1}(1_{(1)})$ has a right inverse y in P (i.e., $xy = 1_P$), and $Ky \subseteq K$, then K is a two-sided ideal and $K = J \cap B$.*

Proof. Let us define

$$K = (p \mapsto p_{(0)}\gamma^{-1}(p_{(1)}))(J). \tag{75}$$

Note that $K \subseteq J$. Therefore, $K\gamma(C) \subseteq J$. On the other hand, for all $p \in J$, $p = p_{(0)}\gamma^{-1}(p_{(1)})\gamma(p_{(2)}) \in K\gamma(C)$. Hence $J = K\gamma(C)$.

Let $b \in B$, $p \in J$, $b' = p_{(0)}\gamma^{-1}(p_{(1)}) \in K$. Then $bb' = bp_{(0)}\gamma^{-1}(p_{(1)}) = (bp)_{(0)}\gamma^{-1}((bp)_{(1)}) \in K$, hence K is a left ideal in B.

Suppose that the element $x = 1_{(0)}\gamma^{-1}(1_{(1)})$ has the right inverse y in P, and $Ky \subseteq K$. As shown above, $K \subseteq B \cap J$. On the other hand, let $b \in B \cap J$. Then $b = \sum_i k_i\gamma(c_i)$, for some $k_i \in K$, $c_i \in C$. It follows that

$$b1_{(0)} \otimes 1_{(1)} = \sum_i k_i\gamma(c_{i(1)}) \otimes c_{i(2)},$$

hence

$$b1_{(0)}\gamma^{-1}(1_{(1)}) = \sum_i k_i\gamma(c_{i(1)})\gamma^{-1}(c_{i(2)}) = \sum_i k_i\varepsilon(c_i),$$

and therefore $b = \sum_i k_i y\varepsilon(c_i) \in K$. Hence $K = J \cap B$ and it follows that K is a two-sided ideal in B. ☐

From the proof of the above lemma we immediately obtain

Corollary 6.3. *Suppose that $P(B)^C_\gamma$ is a cleft C-coalgebra Galois extension. Let K be an ideal in B, and let $J = K\gamma(C)$ be an ideal in P. Moreover suppose that the element $x = 1_{(0)}\gamma^{-1}(1_{(1)}) \in B$ has a right inverse in B. Then $K = J \cap B$.*

Definition 6.4. Let $(P(B)^C, (J_i)_{i\in I})$ be a locally cleft extension. Suppose that, for all $i \in I$, the element $1_{(0)}\gamma_i^{-1}(1_{(1)})$ has a right inverse in B_i. Such a locally cleft extension we shall call *a regular locally cleft extension*.

Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a regular locally cleft extension. Then, by Lemma 5.3, for all $i, j, k \in I$, $P_{ij}^{\mathrm{co}C} = \chi^i_j(B_i) = \chi_{ij}(B)$, $P_{ijk}^{\mathrm{co}C} = \chi^i_{jk}(B_i) = \chi_{ijk}(B)$, etc.

For all $i, j \in I$, $\ker\chi^i_j$ is an ideal and a right C-subcomodule of P_i. Therefore, by Lemma 6.2, $\ker\chi^i_j = \bar{K}^i_j\gamma_i(C)$, where $\bar{K}^i_j = \ker(\chi^i_j)\cap B_i$ is an ideal in B_i. Note that $\ker\pi^i_j = \pi_i(\ker\pi_j) \subseteq \bar{K}^i_j$. Define $K^i_j = \pi^i_j(\bar{K}^i_j)$. Observe that

$$P_{ij}^{\mathrm{co}C} = \chi^i_j(B_i) = B_i/\bar{K}^i_j = \frac{B_i/\ker(\pi^i_j)}{\bar{K}^i_j/\ker(\pi^i_j)}\pi^i_j(B_i)/\pi^i_j(\bar{K}^i_j) = B_{ij}/K^i_j. \tag{76}$$

A locally cleft extension $(P(B)^C, (J_i)_{i\in I})$ is proper if and only if, for all $i, j \in I$, $K^i_j = \{0\}$ (i.e., $\bar{K}^i_j = \ker\pi^i_j$). Note that the properness of a locally cleft extension implies that $B_{ij} = P_{ij}^{\mathrm{co}C}$, and then

$$\ker(\chi^i_{jk})\cap B_i = \ker(\pi^i_{jk}),\ \ \ker(\chi^{ij}_k)\cap B_{ij} = \ker(\pi^{ij}_k),\ \ \text{for all } i, j, k \in I. \tag{77}$$

Indeed,

$$\begin{aligned}\ker(\chi^i_{jk}) = \chi_i(\ker\chi_{jk}) = \chi_i(\ker\chi_j) + \chi_i(\ker\chi_k) &= \ker(\chi^i_j) + \ker(\chi^i_k)\\ &= \ker(\pi^i_j)\gamma_i(C) + \ker(\pi^i_k)\gamma_i(C) = \ker(\pi^i_{jk})\gamma_i(C),\end{aligned}$$

and

$$\ker(\chi^{ij}_k) = \chi^i_j\circ\chi_i(\ker\chi_k) = \chi^i_j(\ker(\pi^i_k)\gamma_i(C)) = \pi^i_j(\ker\pi^i_k)\gamma^i_{ij}(C) = \ker(\pi^{ij}_k)\gamma^i_{ij}(C).$$

Then the relations (77) follow from Corollary 6.3. It follows that, for all $i, j, k \in I$, $B_{ijk} = \pi^{ij}_k(B_{ij})$ is isomorphic to $P_{ijk}^{\mathrm{co}C} = \chi^{ij}_k(B_{ij})$, and can be identified with it. Under this identification,

$$\chi^i_{jk}\Big|_{B_i} = \pi^i_{jk},\ \ \chi^{ij}_k\Big|_{B_{ij}} = \pi^{ij}_k. \tag{78}$$

In what follows, we shall examine conditions for a regular locally cleft extension to be a proper locally cleft extension. This requires the study of ideals K^i_j, $i, j \in I$. We generalise steps of the proof of Proposition 2 [9].

For all $i, j \in I$, let us define an isomorphism (cf. (6))

$$\beta^i_{ij} : P_{ij} \to P_{ij}^{\mathrm{co}C}\otimes C,\ \ p_{ij} \mapsto \theta_{\gamma^i_{ij}}(p_{ij}) = p_{ij(0)}(\gamma^i_{ij})^{-1}(p_{ij(1)})\otimes p_{ij(2)}. \tag{79}$$

Lemma 6.5. *(Cf. the proof of Proposition 2 [9].) Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a regular locally cleft extension. Then, for all $i, j \in I$, $K^i_j = K^j_i$.*

Proof. (Cf. the proof of Proposition 2 [9].) For all $i, j \in I$, define the map

$$\tilde{\phi}_{ji} : B_i \otimes C \to P_{ij}^{\mathrm{co}C} \otimes C, \;\; b_i \otimes c \mapsto \beta_{ij}^j \circ \chi_j^i(b_i \gamma_i(c)). \tag{80}$$

It is easy to see that $\ker \tilde{\phi}_{ji} = \bar{K}_j^i \otimes C$. Consider the maps

$$Q_{ji} : B_{ij} \to P_{ij}^{\mathrm{co}C}, \;\; \pi_j^i(b_i) \mapsto (P_{ij}^{\mathrm{co}C} \otimes \varepsilon) \circ \tilde{\phi}_{ji}(b_i 1_{(0)} \gamma_i^{-1}(1_{(1)}) \otimes 1_{(2)}). \tag{81}$$

Note that, as $\pi_i(\ker \pi_j) \subseteq \bar{K}_j^i$, and $\bar{K}_j^i$ is an ideal, for all $i, j \in I$, the maps Q_{ji} are well defined. Suppose that $b_{ij} \in K_j^i$. There exists an element $b_i \in \bar{K}_j^i$ such that $\pi_j^i(b_i) = b_{ij}$. It follows that $b_i 1_{(0)} \gamma_i^{-1}(1_{(1)}) \otimes 1_{(2)} \in \bar{K}_j^i \otimes C = \ker \tilde{\phi}_{ji}$, hence $Q_{ji}(b_{ij}) = 0$, and so $K_j^i \subseteq \ker Q_{ji}$, for all $i, j \in I$. On the other hand, for all $i, j \in I$ and $b \in B$,

$$\begin{aligned}
Q_{ji}(\pi_{ij}(b))(P_{ij}^{\mathrm{co}C} \otimes \varepsilon) \circ \tilde{\phi}_{ji}(\pi_i(b) 1_{(0)} \gamma_i^{-1}(1_{(1)}) \otimes 1_{(2)}) \\
= (P_{ij}^{\mathrm{co}C} \otimes \varepsilon) \circ \beta_{ij}^j(\chi_{ij}(b)) = (P_{ij}^{\mathrm{co}C} \otimes \varepsilon)(\chi_{ij}(b_{(0)})(\gamma_{ij}^j)^{-1}(b_{(1)}) \otimes b_{(2)}) \\
= \chi_i^j(\pi_j(b) 1_{(0)} \gamma_j^{-1}(1_{(1)})) = \pi_{ij}(b) \pi_i^j(1_{(0)} \gamma_j^{-1}(1_{(1)})) + K_i^j.
\end{aligned}$$

Suppose that, for some element $b_{ij} \in B_{ij}$, $Q_{ji}(b_{ij}) = 0$, i.e., $b_{ij} \pi_j^i(1_{(0)} \gamma_j^{-1}(1_{(1)})) \in K_i^j$. But the element $1_{(0)} \gamma_j^{-1}(1_{(1)})$, by assumption, has a right inverse in B_j, and K_i^j is an ideal in B_{ij}, hence $b_{ij} \in K_i^j$. It follows that $\ker(Q_{ij}) = K_i^j$.

We have proven, that, for all $i, j \in I$, $K_j^i \subseteq K_i^j$, and therefore, for all $i, j \in I$, $K_j^i = K_i^j$. ☐

Let $(P(B)^C, (J_i)_{i \in I})$ be a regular locally cleft extension such that the coalgebra C is flat as a $\mathbb{K}$-module. Recall from the discussion following equation (14) that $P^{\mathbf{c}}$ is naturally an algebra and right C-comodule, and the map $\kappa_P : P \to P^{\mathbf{c}}$, $p \mapsto (\chi_i(p))_{i \in I}$ is an algebra and a right C-comodule isomorphism. It follows that $\kappa_P(B) = (P^{\mathbf{c}})^{\mathrm{co}C}$. Is is clear that

$$\kappa_P(B) \subseteq \breve{B} = \{(b_i)_{i \in I} \in \bigoplus_{i \in I} B_i \mid \forall_{i,j \in I} \; \chi_j^i(b_i) = \chi_i^j(b_j)\}. \tag{82}$$

On the other hand, let $(b_i)_{i \in I} \in \breve{B}$, then, for all $(p_i)_{i \in I} \in P^{\mathbf{c}}$,

$$\rho^C((b_i)_{i \in I}(p_j)_{j \in I}) = (\rho^C(b_i p_i))_{i \in I} = (b_i \rho^C(p_i))_{i \in I} = (b_i)_{i \in I} \rho^C((p_j)_{j \in I}),$$

i.e., $(b_i)_{i \in I} \in (P^{\mathbf{c}})^{\mathrm{co}C}$. It follows that $\breve{B} = (P^{\mathbf{c}})^{\mathrm{co}C} = \kappa_P(B)$.

Suppose that $(B \cap J_i)_{i \in I}$ is a complete covering of B. Then

$$\kappa_P(B) = B^{\mathbf{c}} = \{(b_i)_{i \in I} \in \bigoplus_{i \in I} B_i \mid \forall_{i,j \in I} \; \pi_j^i(b_i) = \pi_i^j(b_j)\}. \tag{83}$$

Let $\mu_{ij} : B_{ij} \to B_{ij}/K^i_j = P^{\mathrm{co}C}_{ij}$ be the canonical surjections. Observe that $\chi^i_j\Big|_{B_i} = \mu_{ij} \circ \pi^i_j$, for all $i, j \in I$. By (82) and (83), for all $(b_i)_{i\in I} \in \bigoplus_{i\in I} B_i$, the condition

$$\pi^i_j(b_i) = \pi^j_i(b_j), \text{ for all } i, j \in I, \tag{84}$$

is equivalent to

$$\mu_{ij}(\pi^i_j(b_i) - \pi^j_i(b_j)) = 0, \text{ for all } i, j \in I. \tag{85}$$

In particular, we have the following

Proposition 6.6. *(cf. Proposition 2, [8].) Let $(P(B)^C, (J_i)_{i\in I})$ be a regular locally cleft extension such that the coalgebra C is flat as a $\mathbb{K}$-module, and $I = \{1, 2\}$. Then $(P(B)^C, (J_i)_{i\in I})$ is a proper locally cleft extension.*

Proof. We prove by contradiction. Suppose that $K^1_2 \neq \{0\}$. Then there exists an element $r \in \bar{K}^1_2$ such that $\pi^1_2(r) \neq 0$. Let $(b_1, b_2) \in B^{\mathsf{c}}$, then $(b_1 + r, b_2) \in \breve{B}$ and $(b_1 + r, b_2) \notin B^{\mathsf{c}}$ which, by the discussion preceding the above proposition, is impossible. □

Suppose that $(P(B)^C, (J_i)_{i\in I})$ is a proper and regular locally cleft extension. Let us define the family of gauge transformations

$$\Xi_{ij} : C \to B_{ij},\ c \mapsto \gamma^i_{ij}(c_{(1)})(\gamma^j_{ij})^{-1}(c_{(2)}), \text{ for all } i, j \in I. \tag{86}$$

The gauge transformations Ξ_{ij}, $i, j \in I$ satisfy the following conditions. For all $i, j, k \in I$, $c \in C$,

$$\Xi_{ii}(c) = \varepsilon(c), \quad \Xi_{ji} = (\Xi_{ij})^{-1}, \tag{87}$$

$$\pi^{ij}_k(\Xi_{ij}(c))\pi^{ik}_j(\Xi_{ik}(c_{(1)}))\pi^{kj}_i(\Xi_{kj}(c_{(2)})). \tag{88}$$

The first two of the above equalities are obvious, to prove the last one observe that, using (78), we obtain, for all $i, j, k \in I$,

$$\begin{aligned}
\pi^{ik}_j(\Xi_{ik}(c_{(1)}))\pi^{kj}_i(\Xi_{kj}(c_{(2)})) &= \chi^{ik}_j(\gamma^i_{ik}(c_{(1)})(\gamma^k_{ik})^{-1}(c_{(2)}))\chi^{kj}_i(\gamma^k_{kj}(c_{(3)})(\gamma^j_{kj})^{-1}(c_{(4)})) \\
&= \chi^i_{kj}(\gamma_i(c_{(1)}))\chi^k_{ij}(\gamma_k^{-1}(c_{(2)}))\chi^k_{ij}(\gamma_k(c_{(3)}))\chi^j_{ik}(\gamma_j^{-1}(c_{(4)})) \\
&= \chi^i_{kj}(\gamma_i(c_{(1)}))\chi^j_{ik}(\gamma_j^{-1}(c_{(2)})) = \chi^{ij}_k(\chi^i_j(\gamma_i(c_{(1)}))\chi^j_i(\gamma_j^{-1}(c_{(2)})) \\
&= \chi^{ij}_k(\gamma^i_{ij}(c_{(1)})(\gamma^j_{ij})^{-1}(c_{(2)})) = \pi^{ij}_k(\Xi_{ij}(c)).
\end{aligned}$$

Suppose that the ground ring $\mathbb{K}$ is a field and $I = \{1, 2\}$. Let $(P(B)^C, (J_i)_{i\in I})$ be a proper locally cleft C-coalgebra Galois extension. By Corollary 5.10, $P(B)^C$ is a C-coalgebra Galois extension if and only if

$$\ker(\chi_1 \otimes_B \chi_1) \cap \ker(\chi_2 \otimes_B \chi_2) = \{0\}. \tag{89}$$

We shall devote the remainder of the present section to proving that, under not very restrictive assumptions, the above condition is always satisfied.

Lemma 6.7. *Let M_1, M_2, M_{12} be $\mathbb{K}$-vector spaces, let $\pi_2^1 : M_1 \to M_{12}$, $\pi_1^2 : M_2 \to M_{12}$ be surjective linear morphisms. Let $M = \{(m, n) \in M_1 \oplus M_2 | \pi_2^1(m) = \pi_1^2(n)\}$, and denote the projections on the summands of the direct sum by $\pi_1 : M \to M_1$, $(m, n) \mapsto m$ and $\pi_2 : M \to M_2$, $(m, n) \mapsto n$. As $\mathbb{K}$ is a field, $\ker \pi_2^1$ and $\ker \pi_1^2$ are direct summands in M_1 and M_2 respectively, i.e., $M_1 = \overline{M}_1 \oplus \ker(\pi_2^1)$, $M_2 = \overline{M}_2 \oplus \ker(\pi_1^2)$, for some subspaces $\overline{M}_i \subseteq M_i$, $i = 1, 2$. Let*

$$\begin{array}{ll} \{m_i\} \textit{ be a basis of } \ker \pi_2^1, & \{n_i\} \textit{ be a basis of } \ker \pi_1^2, \\ \{\bar{m}_i\} \textit{ be a basis of } \overline{M}_1, & \{\bar{n}_i\} \textit{ be a basis of } \overline{M}_2. \end{array}$$

Suppose that $f : \overline{M}_1 \to M_2$ is a linear map such that, for all $m \in \overline{M}_1$, $\pi_2^1(m) = \pi_1^2(f(m))$. Then

the family $\{(0, n_i)\}$ is a basis of $\ker \pi_1$,
the family $\{(m_i, 0)\}$ is a basis of $\ker \pi_2$,
the family $\{(\bar{m}_i, f(\bar{m}_i))\}$ is linearly independent.

Moreover, denote $\overline{M} = Span(\{(\bar{m}_i, f(\bar{m}_i))\})$. Then $M = \overline{M} \oplus \ker(\pi_1) \oplus \ker(\pi_2)$.

Remark. Observe that the map f in Lemma 6.7 is necessarily injective. Note furthermore that the restriction $\bar{\pi}_2^1 : \overline{M}_1 \to M_{12}$ (resp. $\bar{\pi}_1^2 : \overline{M}_2 \to M_{12}$) of the map π_2^1 (resp. π_1^2) is an isomorphism, and, in particular, the map $f = (\bar{\pi}_1^2)^{-1} \circ \bar{\pi}_2^1 : \overline{M}_1 \to M_2$ satisfies the assumptions of the above lemma. Finally, the restriction $\bar{\pi}_{12} : \overline{M} \to M_{12}$ of the map $\pi_2^1 \circ \pi_1$ is an obvious linear isomorphism.

Proof. Suppose that $x \in \overline{M} \cap (\ker(\pi_1) + \ker(\pi_2))$. Then $x = (m, 0) + (0, n) = (m, n)$, for some elements $m \in \ker(\pi_2^1)$, $n \in \ker(\pi_1^2)$. On the other hand, $x \in \mathrm{Span}(\{(\bar{m}_i, f(\bar{m}_i))\})$, hence $x = \sum_i \alpha_i(\bar{m}_i, f(\bar{m}_i)) = (\sum_i \alpha_i \bar{m}_i, f(\sum_i \alpha_i \bar{m}_i))$, for some coefficients $\alpha_i \in \mathbb{K}$. Consequently, $\sum_i \alpha_i \bar{m}_i \in \ker(\pi_2^1)$. As $\mathrm{Span}(\{\bar{m}_i\}) \cap \ker(\pi_2^1) = \{0\}$, we have $x = 0$.

Suppose that $x \in M$. Then there exist unique coefficients $\alpha_i, \beta_i, \gamma_i, \delta_i \in \mathbb{K}$, such that

$$\begin{aligned} x &= \left(\sum_i \alpha_i \bar{m}_i + \sum_j \beta_j m_j, \sum_k \gamma_k \bar{n}_k + \sum_l \delta_l n_l \right) \\ &= \sum_i \alpha_i(\bar{m}_i, f(\bar{m}_i)) + \left(0, \sum_k \gamma_k \bar{n}_k - f\left(\sum_i \alpha_i \bar{m}_i\right)\right) + \sum_j \beta_j(m_j, 0) + \sum_l \delta_l(0, n_l). \end{aligned}$$

Observe that

$$\pi_1^2 \left(\sum_k \gamma_k \bar{n}_k - f\left(\sum_i \alpha_i \bar{m}_i \right) \right) = \pi_1^2 \left(\sum_k \gamma_k \bar{n}_k \right) - \pi_2^1 \left(\sum_i \alpha_i \bar{m}_i \right) = 0.$$

Therefore there exist coefficients $\xi_s \in \mathbb{K}$, such that $\sum_k \gamma_k \bar{n}_k - f(\sum_i \alpha_i \bar{m}_i) = \sum_s \xi_s n_s$. It follows that M is spanned by vectors of the form $(m_i, 0)$, $(0, n_i)$, $(\bar{m}_i, f(\bar{m}_i))$. Their linear independence is obvious. $\square$

Lemma 6.8. *Assume that the ground ring $\mathbb{K}$ is a field, and suppose that $P(B)^C_\gamma$ is a cleft C-coalgebra Galois extension. Let $\{b_i\}$ be a linear basis of B and let $\{h_i\}$ be a linear basis of C. Then*

$$\{b_i\gamma(h_j) \otimes_B \gamma(h_k)\} \tag{90}$$

is a linear basis of $P \otimes_B P$.

Proof. The map

$$F : P \otimes_B P \to P \otimes C, \;\; p \otimes_B p' \mapsto pp'_{(0)}\gamma^{-1}(p'_{(1)}) \otimes p'_{(2)}, \tag{91}$$

is a linear isomorphism (cf. Proposition 2.2), such that, for all i, j, k,

$$F(b_i\gamma(h_j) \otimes_B \gamma(h_k)) = b_i\gamma(h_j) \otimes h_k. \tag{92}$$

Vectors $\{b_i\gamma(h_j) \otimes h_k\}$ form a basis of $P \otimes C$. We conclude that the family (90) is a basis of $P \otimes_B P$. □

Proposition 6.9. *Suppose that the ground ring $\mathbb{K}$ is a field and that $I = \{1, 2\}$. Let $(P(B)^C, (J_i)_{i\in I})$ be a regular locally cleft C-coalgebra Galois extension. Assume that*

$$\gamma_2(C) \ker(\pi^2_1) = \ker(\pi^2_1)\gamma_2(C). \tag{93}$$

Then $P(B)^C$ is a C-coalgebra Galois extension.

Remark. Note that if C is a Hopf algebra and $P_2 = B_2 \otimes C$, then one can choose $\gamma_2 : C \to P_2$, $c \mapsto 1 \otimes c$, and then the condition (93) is automatically satisfied.

Proof. Suppose that $B_1 = \overline{B}_1 \oplus \ker(\pi^1_2)$, $B_2 = \overline{B}_2 \oplus \ker(\pi^2_1)$ and

$\{\bar{x}_i\}$ is a basis of $\overline{B}_1$, $\{\bar{y}_i\}$ is a basis of $\overline{B}_2$, $\{h_i\}$ is a basis of C,
$\{x_i\}$ is a basis of $\ker \pi^1_2$, $\{y_i\}$ is a basis of $\ker \pi^2_1$.

By Proposition 2.2,

$$P_1 = \overline{B}_1\gamma_1(C) \oplus \ker(\pi^1_2)\gamma_1(C), \;\; P_2 = \overline{B}_2\gamma_2(C) \oplus \ker(\pi^2_1)\gamma_2(C).$$

Let $\overline{P}_1 = \overline{B}_1\gamma_1(C)$, $\overline{P}_2 = \overline{B}_2\gamma_2(C)$. By Proposition 6.6, $(P(B)^C, (J_i)_{i\in I})$ is a proper locally cleft extension. Therefore $\ker \chi^1_2 = \ker(\pi^1_2)\gamma_1(C)$ and $\ker \chi^2_1 = \ker(\pi^2_1)\gamma_2(C)$. It follows that

$$P_1 = \overline{P}_1 \oplus \ker(\chi^1_2), \;\; P_2 = \overline{P}_2 \oplus \ker(\chi^2_1), \tag{94}$$

and then,

$$\begin{array}{ll} \{\bar{x}_i\gamma_1(h_j)\} \text{ is a basis of } \overline{P}_1, & \{\bar{y}_i\gamma_2(h_j)\} \text{ is a basis of } \overline{P}_2 \\ \{x_i\gamma_1(h_j)\} \text{ is a basis of } \ker(\chi^1_2), & \{y_i\gamma_2(h_j)\} \text{ is a basis of } \ker(\chi^2_1). \end{array} \tag{95}$$

For all i, let $\check{x}_i$ denote the unique element of $\overline{B}_2$ such that $\pi^1_2(\bar{x}_i) = \pi^2_1(\check{x}_i)$. Similarily, for all i, let $\check{h}_i$ (resp. $\breve{g}_i$) be the unique element of $\overline{P}_2$ (resp $\overline{P}_1$) with the property

$\chi_2^1(\gamma_1(h_i)) = \chi_1^2(\breve{h}_i)$ (resp. $\chi_1^2(\gamma_2(h_i)) = \chi_2^1(\breve{g}_i)$). Let us define the following elements of P:

$$\begin{array}{lll} X_i = \kappa_P^{-1}((x_i, 0)), & Y_j = \kappa_P^{-1}((0, y_j)), & \overline{X}_k = \kappa_P^{-1}((\bar{x}_k, \breve{x}_k)), \\ H_s = \kappa_P^{-1}((\gamma_1(h_s), \breve{h}_s)), & G_t = \kappa_P^{-1}((\breve{g}_t, \gamma_2(h_t))), & \end{array} \tag{96}$$

for all i, j, k, s, t. Note that condition (93) implies that

$$\mathrm{Span}(\{Y_j G_t\}) = \mathrm{Span}(\{G_t Y_j\}). \tag{97}$$

Let us define $\overline{B} = \mathrm{Span}(\{\overline{X}_k\})$, $\overline{P} = \mathrm{Span}(\{\overline{X}_k H_s\})$. It follows immediately from Lemma 6.7 that

$$B = \overline{B} \oplus \ker(\pi_1) \oplus \ker(\pi_2), \quad P = \overline{P} \oplus \ker(\chi_1) \oplus \ker(\chi_2), \tag{98}$$

and

$$\begin{array}{llllll} \{X_i\} & \text{is a basis of} & \ker(\pi_2), & \{X_i H_s\} & \text{is a basis of} & \ker(\chi_2), \\ \{Y_j\} & \text{is a basis of} & \ker(\pi_1), & \{Y_j G_t\} & \text{is a basis of} & \ker(\chi_1), \\ \{\overline{X}_k\} & \text{is a basis of} & \overline{B}, & \{\overline{X}_k H_s\} & \text{is a basis of} & \overline{P}. \end{array} \tag{99}$$

Denote $\overline{P \otimes_B P} = \mathrm{Span}(\{\overline{X}_k H_s \otimes_B H_t\})$. We claim that

$$P \otimes_B P = \overline{P \otimes_B P} \oplus \ker(\chi_1 \otimes_B \chi_1) \oplus \ker(\chi_2 \otimes_B \chi_2), \tag{100}$$

and

$$\begin{array}{lll} \{\overline{X}_k H_s \otimes_B H_t\} & \text{is a basis of} & \overline{P \otimes_B P}, \\ \{X_i H_s \otimes_B H_t\} & \text{is a basis of} & \ker(\chi_2 \otimes_B \chi_2), \\ \{Y_j G_s \otimes_B G_t\} & \text{is a basis of} & \ker(\chi_1 \otimes_B \chi_1). \end{array} \tag{101}$$

First we prove that the above vectors span $P \otimes_B P$. Denote $(P \otimes_B P)' =$ (Span of the vectors (101)). By statements (99), it is obvious that the vectors

1)	$X_i H_s \otimes_B X_j H_t$,	**2)**	$X_i H_s \otimes_B Y_m G_t$,	**3)**	$X_i H_s \otimes_B \overline{X}_k H_t$,
4)	$Y_m G_s \otimes_B X_i H_t$,	**5)**	$Y_m G_s \otimes_B Y_n G_t$,	**6)**	$Y_m G_s \otimes_B \overline{X}_k H_t$,
7)	$\overline{X}_k H_s \otimes_B X_i H_t$,	**8)**	$\overline{X}_k H_s \otimes_B Y_n G_t$,	**9)**	$\overline{X}_k H_s \otimes_B \overline{X}_l H_t$,

for all i, j, k, l, m, n, s, t, span $P \otimes_B P$. Moving B-factors from the right to the left leg in each of the above tensor products, and then using the list (99), we obtain the following results.

The tensor products of types 2) and 4) are simply equal to zero, as

$$\ker(\chi_1)\ker(\pi_2) = \ker(\chi_2)\ker(\pi_1) = \{0\}.$$

The tensor products of types 1), 3), 7) clearly belong to

$$\ker(\chi_2) \otimes_B \mathrm{Span}(\{H_t\}) \subseteq \mathrm{Span}(\{X_i H_s \otimes_B H_t\}) \subseteq (P \otimes_B P)'.$$

The tensor products of types 5) and 8) clearly belong to

$$\ker(\chi_1) \otimes_B \mathrm{Span}(\{G_t\}) \subseteq \mathrm{Span}(\{Y_j G_s \otimes_B G_t\}) \subseteq (P \otimes_B P)'.$$

The tensor products of type 6) belong to

$$\begin{aligned}\ker(\chi_1) \otimes_B \mathrm{Span}(\{H_t\}) &= \mathrm{Span}(\{Y_j G_s\}) \otimes_B \mathrm{Span}(\{H_t\}) \\ &= \mathrm{Span}(\{G_s Y_j\}) \otimes_B \mathrm{Span}(\{H_t\}) \subseteq \mathrm{Span}(\{G_s\}) \otimes_B \ker(\chi_1) \\ &= \mathrm{Span}(\{G_s\}) \otimes_B \mathrm{Span}(\{Y_j G_t\}) \\ &\subseteq \mathrm{Span}(\{Y_j G_s \otimes_B G_t\}) \subseteq (P \otimes_B P)',\end{aligned}$$

where in the second equality we used eq. (97).
The tensor products of type 9) belong to

$$P \otimes_B \mathrm{Span}(\{H_t\})\mathrm{Span}(\{\overline{X}_k H_s \otimes_B H_t\}) \oplus \mathrm{Span}(\{X_i H_s \otimes_B H_t\}) \\ \oplus \mathrm{Span}(\{Y_j G_s \otimes_B H_t\}).$$

In the previous point, we have proven that the last direct summand in the above expression is also contained in $(P \otimes_B P)'$. Therefore tensors of type 9) belong to $(P \otimes_B P)'$. It follows that the tensors (101) span $P \otimes_B P$.

Suppose that the element $z \in \ker(\chi_1 \otimes_B \chi_1) \cap \ker(\chi_2 \otimes_B \chi_2)$. As tensors (101) span $P \otimes_B P$, there exists a family of coefficients $a_{ist}, b_{jst}, c_{kst} \in \mathbb{K}$, such that

$$z = \sum_{i,s,t} a_{ist} X_i H_s \otimes_B H_t + \sum_{j,s,t} b_{jst} Y_j G_s \otimes_B G_t + \sum_{k,s,t} c_{kst} \overline{X}_k H_s \otimes_B H_t. \tag{102}$$

Note that, for all i, j, k, s,

$$\chi_1(X_i) = x_i,\ \chi_1(Y_j) = 0,\ \chi_1(\overline{X}_k) = \bar{x}_k,\ \chi_1(H_s) = \gamma_1(h_s), \\ \chi_2(Y_j) = y_j,\ \chi_2(G_s) = \gamma_2(h_s).$$

It follows that

$$0 = (\chi_1 \otimes_B \chi_1)(z) = \sum_{i,s,t} a_{ist} x_i \gamma_1(h_s) \otimes_B \gamma_1(h_t) + \sum_{k,s,t} c_{kst} \bar{x}_k \gamma_1(h_s) \otimes_B \gamma_1(h_t).$$

By Lemma 6.8, this implies that $a_{ist} = c_{kst} = 0$, for all i, k, s, t. Then

$$0 = (\chi_2 \otimes_B \chi_2)(z) = \sum_{j,s,t} b_{jst} y_j \gamma_2(h_s) \otimes_B \gamma_2(h_t).$$

It follows, by Lemma 6.8, that $b_{jst} = 0$, for all j, s, t, i.e., $z = 0$ and $\ker(\chi_1 \otimes_B \chi_1) \cap \ker(\chi_2 \otimes_B \chi_2) = \{0\}$. By Corollary 5.10, $P(B)^C$ is a C-coalgebra Galois extension. Note that we have also proven that the tensors (101) are linearly independent, and, consequently, they form a basis of $P \otimes_B P$. □

7 Gluing cleft extensions

The quantum geometry situation, which corresponds to the most usual setting for the classical method of constructing principal bundles by patching together trivial principal bundles, is as follows. We are given an algebra B, which has a complete

covering $(K_i)_{i\in I}$, and a coalgebra C. For each of the quotient spaces $B_i = B/K_i$, $i \in I$ (resp. $B_{ij} = B/(K_i + K_j)$, $i,j \in I$), we construct a cleft C-coalgebra Galois extension $P_i(B_i)^C_{\gamma_i}$ (resp. $P_{ij}(B_{ij})^C_{\gamma_{ij}}$). Let us denote by $\pi_i : B \to B_i$, $\pi^i_j : B_i \to B_{ij}$, $i,j \in I$, etc., the canonical surjections. Then we choose surjective algebra and right C-comodule morphisms

$$\chi^i_j : P_i \to P_{ij}, \ \ i,j \in I, \tag{103}$$

such that $\chi^i_j(b_i) = \pi^i_j(b_i)$, for all $i,j \in I$, $b_i \in B_i$, and we use them for gluing (cf. 19)

$$P = \bigoplus_{\chi^i_j} P_i \{(p_i)_{i\in I} \in \bigoplus_{i\in I} P_i \mid \forall_{i,j\in I}\ \chi^i_j(p_i) = \chi^j_i(p_j)\}. \tag{104}$$

If the coalgebra C is flat as a $\mathbb{K}$-module then (cf. the discussion in Section 4) P is naturally a right C-comodule. Then, for each $n \in I$, we define the algebra and right C-comodule map

$$\chi_n : P \to P_n, \ \ (p_i)_{i\in I} \mapsto p_n. \tag{105}$$

If all the maps χ_i, $i \in I$, are surjective then $(P(B)^C, (\ker \chi_i)_{i\in I})$ is a proper locally cleft C-coalgebra Galois extension. Moreover, for all $i \in I$, $P_i \simeq P/\ker(\chi_i)$. The following lemma gives necessary and sufficient conditions for the maps χ_i, $i \in I$, to be surjective.

Lemma 7.1. *Suppose that, for each $i \in I$, the element $1_{(0)}\gamma_i^{-1}(1_{(1)}) \in B_i$ has a right inverse in B_i. The algebra and right C-comodule maps (103) satisfy the condition $\chi^i_j(b_i) = \pi^i_j(b_i)$, for all $i,j \in I$, $b_i \in B_i$, if and only if there exists a family of convolution invertible maps (gauge transformations) $\Gamma^i_j : C \to B_{ij}$, $i,j \in I$, such that*

$$\chi^i_j(p_i) = \pi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)}))\Gamma^i_j(p_{i(2)})\gamma_{ij}(p_{i(3)}), \tag{106}$$

for all $i,j \in I$, $p_i \in P_i$. Furthermore, assume that the coalgebra C is flat as a $\mathbb{K}$-module. Let $I = \{1, 2, \ldots, N\}$, and suppose that either $N \leq 3$, or the algebra B and its complete covering $(K_i)_{i\in I}$ satisfy the condition (cf. eq. (25)),

$$\bigcap_{1\leq j\leq i} (\ker \pi^{k+1}_j + \ker \pi^{k+1}_{i+1}) = \left(\bigcap_{1\leq j\leq i} \ker \pi^{k+1}_j \right) + \ker \pi^{k+1}_{i+1}, \tag{107}$$

for all $1 \leq k < N$ and $1 \leq i < k$. Define gauge transformations $\Xi_{ij} : C \to B_{ij}$, $c \mapsto \Gamma^i_j(c_{(1)})(\Gamma^j_i)^{-1}(c_{(2)})$, $i,j \in I$ (cf. eq. (86)). Then the maps (105) are surjective if and only if the condition (88) is satisfied:

$$\pi^{ij}_k(\Xi_{ij}(c))\pi^{ik}_j(\Xi_{ik}(c_{(1)}))\pi^{kj}_i(\Xi_{kj}(c_{(2)})). \tag{108}$$

Remark. It is clear that, while maps Γ^i_j, $i,j \in I$, define surjections χ^i_j, the space $P = \bigoplus_{\chi^i_j} P_i$ is fully defined by the maps Ξ_{ij}, $i,j \in I$. Indeed, for all $i,j \in I$, $p_i \in P_i$, $p_j \in P_j$, the condition $\chi^i_j(p_i) = \chi^j_i(p_j)$ is equivalent to

$$\pi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)}))\Xi_{ij}(p_{i(2)}) \otimes p_{i(3)} = \pi^j_i(p_{j(0)}\gamma_j^{-1}(p_{j(1)})) \otimes p_{j(2)}. \tag{109}$$

Proof. Suppose that the algebra and right C-comodule maps (103) satisfy the condition $\chi^i_j(b_i) = \pi^i_j(b_i)$, for all $i, j \in I$, $b_i \in B_i$. Then, for all $p_i \in P_i$,

$$\chi^i_j(p_i) = \chi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)})\gamma_i(p_{i(2)}))$$
$$= \pi^i_j(p_{i(0)}\gamma_i^{-1}(p_{i(1)}))\chi^i_j(\gamma_i(p_{i(2)}))\gamma_{ij}^{-1}(p_{i(3)})\gamma_{ij}(p_{i(4)})).$$

Defining the maps $\Gamma^i_j(c) = \chi^i_j(\gamma_i(c_{(1)}))\gamma_{ij}^{-1}(c_{(2)})$, $i, j \in I$ yields eq. (106). Conversely, let the maps (103) have the form (106). Then, for all $i, j \in I$, $b_i \in B_i$,

$$\chi^i_j(b_i)\pi^i_j(b_i)\pi^i_j(1_{(0)}\gamma_i^{-1}(1_{(1)}))\Gamma^i_j(1_{(2)})\gamma_{ij}(1_{(3)}) = \pi^i_j(b_i)\chi^i_j(1) = \pi^i_j(b_i).$$

Assume that the coalgebra C is flat as a $\mathbb{K}$-module. We will check that the maps (103) satisfy the assumptions of Proposition 4.2, which in turn will prove that the maps (105) are surjective.

Note that

$$\ker \chi^i_j = \ker(\pi^i_j)\gamma_i(C), \text{ for all } i, j \in I. \tag{110}$$

Indeed, by (6), $\chi^i_j = \theta^{-1}_{\gamma^i_{ij}} \circ (\pi^i_j \otimes C) \circ \theta_{\gamma_i}$, which means that $\ker(\chi^i_j) = \theta^{-1}_{\gamma_i}(\ker(\pi^i_j \otimes C)) = \theta^{-1}_{\gamma_i}(\ker(\pi^i_j) \otimes C) = \ker(\pi^i_j)\gamma_i(C)$, where $\gamma^i_{ij} = \chi^i_j \circ \gamma_i$, and the second equality follows from the flatness of C. Observe that (condition (21) of Proposition 4.2)

$$\chi^i_j(\ker \chi^i_k) = \chi^j_i(\ker \chi^j_k), \text{ for all } i, j, k \in I. \tag{111}$$

Indeed, for all $i, j, k \in I$, $\pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k)$, and then, as $\mathbb{K}$-modules $\pi^i_j(\ker \pi^i_k)$ are ideals, for all $c \in C$, $\pi^i_j(\ker \pi^i_k)\Xi_{ij}(c_{(1)}) \otimes c_{(2)} \subseteq \pi^i_j(\ker \pi^i_k) \otimes C = \pi^j_i(\ker \pi^j_k) \otimes C$. Consequently, $\pi^i_j(\ker \pi^i_k)\gamma^i_{ij}(C) \subseteq \pi^j_i(\ker \pi^j_k)\gamma^j_{ij}(C)$. Furthermore by (110), $\chi^i_j(\ker \chi^i_k) = \pi^i_j(\ker \pi^i_k)\gamma^i_{ij}(C)$, hence it follows that, for all $i, j, k \in I$, $\chi^i_j(\ker \chi^i_k) \subseteq \chi^j_i(\ker \chi^j_k)$.

For each $i, j, k \in I$, the map

$$W^i_{jk} : P_i/(\ker(\chi^i_j) + \ker(\chi^i_k)) \to B_{ijk} \otimes C,$$
$$p_i + \ker(\chi^i_j) + \ker(\chi^i_k) \mapsto \pi^i_{jk}(p_{i(0)}\gamma_i^{-1}(p_{i(1)})) \otimes p_{i(2)}, \tag{112}$$

is an isomorphism. Indeed, since $\ker(\chi^i_j) + \ker(\chi^i_k) = \ker(\pi^i_{jk})\gamma_i(C)$, W^i_{jk} is well defined. It is also obviously surjective. Moreover, suppose that $W^i_{jk}(p_i + \ker(\chi^i_j) + \ker(\chi^i_k)) = 0$. This means that $\theta_{\gamma_i}(p_i) \in \ker(\pi^i_{jk} \otimes C) = \ker(\pi^i_{jk}) \otimes C$, hence $p_i \in \ker(\chi^i_j) + \ker(\chi^i_k)$. Note that, for all $b_i \in B_i$ and $c \in C$,

$$(W^i_{jk})^{-1}(\pi^i_{jk}(b_i) \otimes C) = b_i\gamma_i(c) + \ker(\chi^i_j) + \ker(\chi^i_k).$$

Suppose that the maps

$$\phi^k_{ij} : P_j/(\ker \chi^j_i + \ker \chi^j_k) \to P_i/(\ker \chi^i_j + \ker \chi^i_k), \quad i, j, k \in I$$

are the isomorphisms (23), i.e., for all $p_j \in P_j$,

$$\phi_{ij}^k(p_j + \ker(\chi_i^j) + \ker(\chi_k^j)) = p_i + \ker(\chi_j^i) + \ker(\chi_k^i),$$

where p_i is any element of P_i such that $\chi_j^i(p_i) = \chi_i^j(p_j)$ (cf. Remark after Proposition 4.2). For each $i, j, k \in I$, let us define the isomorphisms

$$\bar{\phi}_{ij}^k = W_{jk}^i \circ \phi_{ij}^k \circ (W_{ik}^j)^{-1} : B_{ijk} \otimes C \to B_{ijk} \otimes C. \tag{113}$$

It is easy to see that explicitly, for all $b_{ijk} \in B_{ijk}$, $c \in C$,

$$\bar{\phi}_{ij}^k(b_{ijk} \otimes c) = b_{ijk}\pi_k^{ij}(\Xi_{ji}(c_{(1)})) \otimes c_{(2)}. \tag{114}$$

Clearly, the condition (24) is equivalent to

$$\bar{\phi}_{ik}^j = \bar{\phi}_{ij}^k \circ \bar{\phi}_{jk}^i, \text{ for all } i, j, k \in I, \tag{115}$$

and this in turn is, by eq. (114), equivalent to the condition (108).

Finally, in view of the flatness of C and the condition (107) we obtain, for all $1 \leq k < N$ and $1 \leq i < k$,

$$\begin{aligned}
\bigcap_{1\leq j\leq i} (\ker \chi_j^{k+1} + \ker \chi_{i+1}^{k+1}) &= \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} \theta_{\gamma_{k+1}}(\ker \chi_j^{k+1} + \ker \chi_{i+1}^{k+1})\right) \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} (\ker(\pi_{j,i+1}^{k+1}) \otimes C)\right) = \theta_{\gamma_{k+1}}^{-1}\left(\ker\left(\bigoplus_{1\leq j\leq i} \pi_{j,i+1}^{k+1} \otimes C\right)\right) \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\ker\left(\bigoplus_{1\leq j\leq i} \pi_{j,i+1}^{k+1}\right) \otimes C\right) = \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} (\ker \pi_j^{k+1} + \ker \pi_{i+1}^{k+1}) \otimes C\right) \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\bigcap_{1\leq j\leq i} (\ker \pi_j^{k+1}) \otimes C\right) + \ker \chi_{i+1}^{k+1} \\
&= \theta_{\gamma_{k+1}}^{-1}\left(\ker\left(\bigoplus_{1\leq j\leq i} \pi_j^{k+1} \otimes C\right)\right) + \ker \chi_{i+1}^{k+1} = \bigcap_{1\leq j\leq i} \ker(\chi_j^{k+1}) + \ker(\chi_{i+1}^{k+1}).
\end{aligned}$$

Thus all the assumptions of Proposition 4.2 are satisfied, and hence the maps (105) are surjective. □

8 Example: The quantum lens spaces

It was shown in [13] that by gluing two quantum discs, D_p and D_q, one can obtain the quantum 2-sphere S^2_{pq}, and that the universal C^* algebra of functions on S_{pq}

is isomorphic to equatorial or latitudinal Podleś spheres ([20]). In [9] a quantum sphere S^3_{pq} was obtained, by gluing quantum solid tori (cf. Subsection 8.3) $D_p \times S^1$ and $D_q \times S^1$, as an example of a locally trivial $U(1)$-quantum principal bundle with the base space S^2_{pq}. It was also shown that $\vartheta(S^3_{pq})(\vartheta(S^2_{pq}))^{\vartheta(U(1))}$ is a principal Hopf-Galois extension.

As an illustration of methods described earlier in this chapter, we will construct a locally cleft Hopf Galois extension of $\vartheta(S^2_{pq})$ by gluing two quantum solid tori (Subsection 8.3) $D_p \times_\theta S^1$ and $D_q \times_\theta S^1$, obtaining this way quantum lens spaces $L^{p,q,\theta}_\beta$ of charge β, for all $\beta \in \mathbb{Z}$. As a special case, for $\beta = 1$, this gives the Heegaard quantum sphere ([2]).

Another example of a construction of quantum lens spaces by gluing two quantum solid tori of type $D \times_\theta S^1$ can be found in [17].

8.1 The quantum unit disc

A two-parameter family of quantum unit discs was defined in [15]. Here we consider the one parameter subfamily studied in [14]. We start with a coordinate $*$-algebra $\vartheta(D_p)$ generated by x and the relation

$$x^*x - pxx^* = 1 - p, \quad 0 < p < 1. \tag{116}$$

The spectrum of xx^* (in any C^*-algebra completion) is

$$\sigma(xx^*) = \{1 - p^n \mid n = 0, 1, 2, \dots\} \cup \{1\}, \tag{117}$$

i.e. $xx^* \leq 1$, where $\leq$ is understood as an order relation between positive operators. This justifies the name 'unit disc'. Furthermore, this relation means also that $\|x\| = 1$ in any C^* completion of $\vartheta(D_p)$ (Theorem 2.1.1 [18]).

Observe that (116) has the following useful symmetry. Let x_- be the generator of $\vartheta(D_{p^{-1}})$, (where we consciously abuse the notation as $p^{-1} \geq 1$), i.e.,

$$x_-^* x_- - p^{-1} x_- x_-^* = 1 - p^{-1}.$$

Then assignment $x \mapsto x_-^*$ can be extended to a $*$-algebra isomorphism

$$\kappa_p : \vartheta(D_p) \to \vartheta(D_{p^{-1}}). \tag{118}$$

The coordinate algebra $\vartheta(D_p)$ can be completed to the C^*-algebra $C(D_p)$ with the norm

$$\|a\| = \sup_\varrho \|a\|_\varrho, \quad a \in \vartheta(D_p), \tag{119}$$

where the supremum is taken over all bounded representations $\varrho : \vartheta(D_p) \to \mathbf{B}(\mathcal{H}_\varrho)$ of $\vartheta(D_p)$ and $\|\cdot\|_\varrho$ denotes the operator norm in the representation ϱ. The C^* algebra $C(D_p)$ is called a *universal enveloping algebra* of $\vartheta(D_p)$. Note that the norm (119) is well defined because $\|x\|_\varrho = 1$ for all ϱ.

Irreducible bounded representations of $\vartheta(D_p)$ are unitarily equivalent to one of the following representations.

1. For all $0 \leq \phi < 2\pi$, there is a one dimensional representation $\varrho_\phi : \vartheta(D_p) \to \mathbb{C}$,

$$\varrho_\phi(x) = e^{i\phi}, \ \ \varrho_\phi(x^*) = e^{-i\phi}. \tag{120}$$

2. There is also an infinitely dimensional representation $\varrho_\infty : \vartheta(D_p) \to \mathbf{B}(\mathcal{H}_\infty)$, where $\mathcal{H}_\infty$ is generated by orthonormal vectors Ψ_n, $n = 0, 1, 2, \ldots$, and

$$\varrho_\infty(x)\Psi_n = \sqrt{1 - p^{n+1}}\Psi_{n+1}, \ \ \varrho_\infty(x^*)\Psi_{n+1} = \sqrt{1 - p^{n+1}}\Psi_n, \ \ \varrho_\infty(x^*)\Psi_0 = 0. \tag{121}$$

The infinite dimensional representation ϱ_∞ is faithful. Nonfaithful representations correspond to subsets of quantum discs. In particular, one-dimensional representations are also characters, that is, they correspond to the classical points of the quantum space. In the case of the quantum disc, one-dimensional representations describe the classical unit circle.

Let us adopt notational convention $x^{-n} \equiv (x^*)^n$. It follows from (116) that $(1 - xx^*)x = px(1 - xx^*)$, and, therefore,

$$(1 - xx^*)^n x^m = p^{mn} x^m (1 - xx^*)^n, \ \ n \in \mathbb{N}, m \in \mathbb{Z}. \tag{122}$$

Moreover, it is easy to prove by induction that, for all $n \geq 0$,

$$(x^*)^n x^n = 1 + \sum_{m=1}^{n} (-1)^m p^{nm - \frac{m(m-1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_{p^{-1}} (1 - xx^*)^m, \tag{123a}$$

$$x^n (x^*)^n = 1 + \sum_{m=1}^{n} (-1)^m p^{-nm + \frac{m(m+1)}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_{p} (1 - xx^*)^m, \tag{123b}$$

where we used the standard p-deformed binomial coefficients

$$\begin{bmatrix} n \\ m \end{bmatrix}_p \frac{[n]_p!}{[m]_p![n-m]_p!},$$
$$[n]_p! = [1]_p[2]_p \ldots [n-1]_p[n]_p \text{ for } n \in \mathbb{N}, \ [0]_p! = 1,$$
$$[n]_p = 1 + p + p^2 + \ldots + p^{n-2} + p^{n-1} \text{ for } n \in \mathbb{N}, \ [0]_p = 0. \tag{124}$$

Note that equation (123b) follows from (123a) by the application of the isomorphism (118).

It is now obvious that, for all $n, m \in \mathbb{Z}$,

$$x^n x^m = x^{n+m}(1 + Q^p_{n;m}(1 - xx^*)), \tag{125}$$

where $Q^p_{n;m}$ is a polynomial of degree at most $\min\{|m|, |n|\}$ and such that $Q^p_{n;m}(0) = 0$. For example, if $m \geq n \geq 0$, then $x^{-n}x^m = (x^{-n}x^n)x^{m-n}$, and then use (123a) and (122).

As elements $x^n x^m$, $n, m \in \mathbb{Z}$ obviously span $\vartheta(D_p)$ as a vector space, (125) means that also elements of the form

$$x^n(1 - xx^*)^m, \ \ n \in \mathbb{Z}, \ m \in \mathbb{N}_0, \tag{126}$$

span $\vartheta(D_p)$. In fact, using the infinite dimensional representation (121) one can prove that the above family forms a basis of $\vartheta(D_p)$. Indeed, suppose that

$$\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} A_{nm}x^n(1-xx^*)^m = 0,$$

for some coefficients $A_{mn} \in \mathbb{C}$, only a finite number of which are different from zero. Therefore, for any $k \geq 0$,

$$\begin{aligned}
0 = \varrho_\infty(\sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} A_{nm}x^n(1-xx^*)^m)\Psi_k &= \sum_{n=-\infty}^{\infty}\sum_{m=0}^{\infty} p^{mk}A_{nm}\varrho_\infty(x^n)\Psi_k \\
&= \sum_{m=0}^{\infty} p^{mk}(\sum_{n=1}^{k} A_{-nm}\sqrt{1-p^k}\ldots\sqrt{1-p^{k-n+1}}\Psi_{k-n} + A_{0m}\Psi_k \\
&\qquad + \sum_{n=1}^{\infty} A_{nm}\sqrt{1-p^{k+1}}\ldots\sqrt{1-p^{k+n}}\Psi_{k+n}.)
\end{aligned}$$

As vectors Ψ_k are linearly independent and $p<1$,

$$\sum_{m=0}^{\infty} p^{mk}A_{nm} = 0 \text{ for all } k \geq 0,\ -k \leq n,$$

i.e., for each $n \in \mathbb{Z}$, the polynomial $\sum_{m=0}^{\infty} A_{nm}t^m$ has an infinite number of distinct roots p^k, $k \geq 0$ if $n \geq 0$ or $k \geq -n$ otherwise. This is possible only if $A_{nm} = 0$ for all $n \in \mathbb{Z}$, $m \geq 0$. In addition, we have proven that the representation (121) is faithful.

8.2 The quantum torus

Quantum torus was defined in [21]. The coordinate algebra of the quantum torus $\vartheta(T_\phi)$, $\phi \in [0, 2\pi)$, is generated by unitary elements U, V which satisfy the following commutation relations

$$UV = e^{i\phi}VU,\ \ UV^* = e^{-i\phi}V^*U. \tag{127}$$

Obviously, the elements

$$V^nU^m, \quad n, m \in \mathbb{Z}, \tag{128}$$

form a basis of $\vartheta(T_\phi)$. The coordinate algebra $\vartheta(T_\phi)$ can be completed to the enveloping C^*-algebra $C(T_\phi)$ using representations of $\vartheta(T_\phi)$. The representation theory of $\vartheta(T_\phi)$ depends on whether ϕ is a rational or irrational multiple of 2π.

Suppose that $\phi = 2\pi\frac{M}{N}$, where $M, N \in \mathbb{N}$, $M < N$, and M and N are relatively prime. Then U^N and V^N are central in $\vartheta(T_\phi)$, and we can classify irreducible representations of $\vartheta(T_\phi)$ according to their eigenvalues. It turns out that irreducible representations of $\vartheta(T_\phi)$ include the ones isomorphic to one of the representations $\varrho_{\alpha\beta} : \vartheta(T_\phi) \to \mathbf{B}(\mathcal{H}^{\alpha\beta})$, $\alpha, \beta \in [0, 2\pi)$, where $\mathcal{H}^{\alpha\beta}$ is spanned by orthonormal vectors $\Psi_n^{\alpha\beta}$, $n \in \mathbb{Z}_N$, and

$$\varrho_{\alpha\beta}(U^{\pm 1})\Psi_n^{\alpha\beta} = e^{\pm i\frac{\alpha}{N}}e^{\pm 2\pi i n\frac{M}{N}}\Psi_n^{\alpha\beta}, \quad \varrho_{\alpha\beta}(V^{\pm 1})\Psi_n^{\alpha\beta} = e^{\pm i\frac{\beta}{N}}\Psi_{n\pm 1}^{\alpha\beta}. \tag{129}$$

On the other hand, if ϕ is an irrational multiple of 2π, then irreducible representations include the representations unitarily isomorphic to one of the $\varrho_\alpha : \vartheta(T_\phi) \to \mathbf{B}(\mathcal{H}^\alpha)$, $\alpha \in [0, 2\pi)$, where $\mathcal{H}^\alpha$ is the closure of the linear span of the family of orthonormal vectors Ψ_n^α, $n \in \mathbb{Z}$, and

$$\varrho_\alpha(U^{\pm 1})\Psi_n^\alpha = e^{\pm i\alpha} e^{\pm in\phi} \Psi_n^\alpha, \quad \varrho_\alpha(V^{\pm 1})\Psi_n^\alpha = \Psi_{n\pm 1}^\alpha. \tag{130}$$

Note, that the quantum torus is not a type I C^*-algebra, therefore (cf. the discussion at the end of the chapter 3 in [1]) its irreducible representations cannot be explicitly listed.

Representations (130) are faithful. Representations (129) are not faithful. However, choose two sequences $(\alpha_n)_{n\in\mathbb{N}}$, $(\beta_n)_{n\in\mathbb{N}}$, such that $\alpha_n, \beta_m \in [0, 2\pi)$, $m, n \in \mathbb{N}$ and $\alpha_n \neq \alpha_m$, $\beta_n \neq \beta_m$ if $n \neq m$. Then the representation

$$\varrho^{(\alpha_n)_{n\in\mathbb{N}}(\beta_n)_{n\in\mathbb{N}}} = \bigoplus_{m,n\in\mathbb{N}} \varrho^{\alpha_m\beta_n} : \vartheta(T_\phi) \longrightarrow \bigoplus_{m,n\in\mathbb{N}} \mathbf{B}(\mathcal{H}^{\alpha_m\beta_n}) \tag{131}$$

is faithful. Indeed, suppose that for some coefficients $A_{mn} \in \mathbb{C}$, $m, n \in \mathbb{Z}$, a finitely many of which are different from zero, $\varrho^{(\alpha_n)_{n\in\mathbb{N}}(\beta_n)_{n\in\mathbb{N}}}(\sum_{m,n\in\mathbb{Z}} A_{mn}V^mU^n) = 0$, i.e., for all $k, l \in \mathbb{Z}$ and $s \in \mathbb{Z}_N$,

$$0 = \varrho^{\alpha_k\beta_l}(\sum_{m,n\in\mathbb{Z}} A_{mn}V^mU^n)\Psi_s^{\alpha_k\beta_l} = \sum_{m,n\in\mathbb{Z}} A_{mn} e^{in\frac{\alpha_k}{N}} e^{2\pi is\frac{M}{N}} e^{im\frac{\beta_l}{N}} \Psi^{\alpha_k\beta_l}_{s+[m]_N}.$$

Then, for all $s \in \mathbb{Z}_N$, $m \in \{0, 1, 2, \ldots, N-1\}$,

$$\sum_{j,n\in\mathbb{Z}} A_{(m+jN)n} e^{in\frac{\alpha_k}{N}} e^{i\beta_l j} = 0 \text{ for all } k, l \in \mathbb{N}.$$

This, by the argument on the number of distinct roots of a finite polynomial, implies that $A_{mn} = 0$, for all $m, n \in \mathbb{Z}$.

8.3 The quantum solid torus

The quantum solid torus is an example of a cleft Hopf-Galois extension.

Denote by h the unitary and central generator of the coordinate algebra of the unit circle $\vartheta(S^1)$.

Solid torus is the Cartesian product $D \times S^1$ of the unit disc and the unit circle. Therefore, one can define a coordinate algebra of quantum torus as the tensor product $\vartheta(D_p) \otimes \vartheta(S^1)$ of the coordinate algebra of the quantum unit disc eq. (116) and the coordinate algebra of the unit circle. One can introduce a further quantisation parameter by making the tensor product noncommutative, i.e., by requesting that the subalgebras $\vartheta(D_p) \otimes 1$ and $1 \otimes \vartheta(S^1)$ do not mutually commute. We identify the generators $x \otimes 1$ and $1 \otimes h$ with x and h respectively. The coordinate algebra of the quantum solid torus $\vartheta(D_p \times_\theta S^1)$, $0 < p < 1$, $0 \leq \theta < 2\pi$, is generated as a *-algebra by x and h, subject to the relations

$$\begin{aligned} hh^* = 1 = h^*h, \;\; x^*x - pxx^* = 1 - p, \\ hx = e^{i\theta}xh, \;\; hx^* = e^{-i\theta}x^*h. \end{aligned} \tag{132}$$

The linear basis of $\vartheta(D_p \times_\theta S^1)$ consists of the elements of the form

$$(1 - xx^*)^k x^m h^n, \quad k \in \mathbb{N}_0,\ m, n \in \mathbb{Z}. \tag{133}$$

Observe that there exists a surjective *-algebra morphism

$$\pi_\partial : \vartheta(D_p \times_\theta S^1) \to \vartheta(T_\theta),\ \pi_\partial(h) = U,\ \pi_\partial(x) = V, \tag{134}$$

of the quantum solid torus onto the quantum torus (eq. (127)), i.e., the quantum torus is the boundary of the quantum solid torus.

Enveloping C^*-algebra $C(D_p \times_\theta S^1)$ can be obtained from $\vartheta(D_p \times_\theta S^1)$ using C^* representations. Irreducible representations of $\vartheta(D_p \times_\theta S^1)$ include those unitarily isomorphic either to the representation obtained by composing one of the irreducible representations of the quantum torus $\vartheta(T_\theta)$ with the map π_∂ as well one of the representations $\varrho^\alpha_{p,\theta} : \vartheta(D_p \times_\theta S^1) \to \mathbf{B}(\mathcal{H}^\alpha_{p,\theta})$, where $\mathcal{H}^\alpha_{p,\theta}$ is generated by orthonormal vectors Ψ_n, $n \in \mathbb{N}_0$, and

$$\begin{aligned} \varrho^\alpha_{p,\theta}(x)\Psi_n = \sqrt{1 - p^{n+1}}\Psi_{n+1},\ \ \varrho^\alpha_{p,\theta}(x^*)\Psi_n = \sqrt{1 - p^n}\Psi_{n-1} \text{ if } n > 0, \\ \varrho^\alpha_{p,\theta}(x^*)\Psi_0 = 0,\ \ \varrho^\alpha_{p,\theta}(h^{\pm 1})\Psi_n = e^{\pm i(\alpha + n\theta)}\Psi_n. \end{aligned} \tag{135}$$

If θ is irrational then the representation (135) is faithful. If θ is rational, then for any sequence $(\alpha_n)_{n\in\mathbb{N}}$, such that $0 \le \alpha_n < 2\pi$ and $\alpha_i \ne \alpha_j$ if $i \ne j$, for all $i, j, n \in \mathbb{N}$, the representation

$$\bigoplus_{n\in\mathbb{N}} \varrho^{\alpha_n}_{p,\theta} : \vartheta(D_p \times_\theta S^1) \to \mathbf{B}(\bigoplus_{n\in\mathbb{N}} \mathcal{H}^{\alpha_n}_{p,\theta})$$

is faithful.

Denote by $H = \vartheta(U(1))$ the coordinate algebra of $U(1)$ and let u be the unitary generator of H. The algebra $\vartheta(D_p \times_\theta S^1)$ is clearly a right H-comodule $*$-algebra, with the coaction defined on the generators by

$$\rho^H(x) = x \otimes 1,\ \ \rho^H(h) = h \otimes u. \tag{136}$$

It is easy to see that, $\vartheta(D_p \times_\theta S^1)^{\mathrm{co}H} = \vartheta(D_p)$ and $\vartheta(D_p \times_\theta S^1)(\vartheta(D_p))^H_{\gamma_T}$ is a cleft H-Hopf Galois extension, where, for all $n \in \mathbb{Z}$,

$$\begin{aligned} \gamma_T : H \to \vartheta(D_p \times_\theta S^1),\ \ u^n \mapsto h^n, \\ \gamma_T^{-1} : H \to \vartheta(D_p \times_\theta S^1),\ \ u^n \mapsto h^{-n}, \end{aligned} \tag{137}$$

$$\tag{138}$$

are the cleaving map and its convolution inverse, respectively.

8.4 Gluing of two quantum solid tori

Let $H = \vartheta(U(1))$ be the Hopf algebra generated by a unitary and group-like element u.

Let the deformation parameters $p, q \in (0,1)$, $\theta, \theta', \theta'' \in \mathbb{R}$. We define $P_1 = \vartheta(D_p \times_\theta S^1)$, $P_2 = \vartheta(D_q \times_{\theta'} S^1)$ (Subsection 8.3), $P_{12} = \vartheta(T_{\theta''})$ (Subsection 8.2). A $*$-algebra P_1 is generated by the elements x, h, which satisfy relations (132), and it is a right H-comodule algebra with the coaction defined by (136). The corresponding generators of P_2, y and g, satisfy the relations

$$\begin{aligned} gg^* = 1 = g^*g, \ \ y^*y - qyy^* = 1 - q, \\ gy = e^{i\theta'}yg, \ \ gy^* = e^{-i\theta'}y^*g. \end{aligned} \tag{139}$$

P_2 is a right H-comodule $*$-algebra with the coaction defined on generators by $\rho^H(y) = y \otimes 1$, $\rho^H(g) = g \otimes u$. Finally, P_{12} is a right H-comodule $*$-algebra generated by unitary elements U and V satisfying $UV = e^{i\theta''}VU$, with the right H-coaction defined by the relations $\rho^H(V) = V \otimes 1$, $\rho^H(U) = U \otimes u$. Note that $B_1 = P_1^{\mathrm{co}H} \simeq \vartheta(D_p)$ (see Section 8.1), is generated as a $*$-algebra by x, $B_2 = P_2^{\mathrm{co}H} \simeq \vartheta(D_q)$ is generated by y and $B_{12} = P_{12} \simeq \vartheta(S^1)$ is generated by V. Let the algebra surjections $\pi_2^1 : B_1 \to B_{12}$, $\pi_1^2 : B_2 \to B_{12}$ be defined on generators by

$$\pi_2^1(x) = V, \ \ \pi_1^2(y) = V. \tag{140}$$

We define cleaving maps by

$$\gamma_1^{\pm 1}(u^n) = h^{\pm n}, \ \ \gamma_2^{\pm 1}(u^n) = g^{\pm n}, \ \ \gamma_{12}^{\pm 1}(u^n) = U^{\pm n}, \quad \text{for all } n \in \mathbb{Z}. \tag{141}$$

Then $P_1(B_1)^H_{\gamma_1}$, $P_2(B_2)^H_{\gamma_2}$, $P_{12}(B_{12})^H_{\gamma_{12}}$ are cleft extensions. By Lemma 7.1, in order to define gluing surjections (103), $\chi_2^1 : P_1 \to P_{12}$, $\chi_1^2 : P_2 \to P_{12}$, we need to find appropriate convolution invertible maps $\Gamma_2^1, \Gamma_1^2 : H \to B_{12}$. To fix the notation, without losing the generality, we shall only consider Γ_1^2.

For all $n \in \mathbb{Z}$, $\Gamma_1^2(u^n)$ and $(\Gamma_1^2)^{-1}(u^n)$ are Laurent polynomials in V such that $\Gamma_1^2(u^n)(\Gamma_1^2)^{-1}(u^n) = 1$. By the standard argument about degree counting, this implies that

$$(\Gamma_1^2)^{\pm 1}(u^n) = \mu(n)^{\pm 1} V^{\pm \nu(n)}, \text{ where } \mu : \mathbb{Z} \to \mathbb{C} \setminus \{0\}, \ \nu : \mathbb{Z} \to \mathbb{Z}. \tag{142}$$

The map χ_1^2 must be algebraic, hence in paricular,

$$\chi_1^2(y^m g^n y^k g^l) = \chi_1^2(y^m g^n)\chi_1^2(y^k g^l), \text{ for all } m, n, k, l \in \mathbb{Z}.$$

Substituting (140), (141), (142) and (106) yields

$$\chi_1^2(y^m g^n y^k g^l) = \chi_1^2(e^{ink\theta'} y^m y^k g^{n+l}) = e^{ink\theta'} \mu(n+l) V^{m+k+\nu(n+l)} U^{n+l},$$

and

$$\begin{aligned} \chi_1^2(y^m g^n)\chi_1^2(y^k g^l) &= \mu(n)\mu(l) V^{m+\nu(n)} U^n V^{k+\nu(l)} U^l \\ &= e^{i\theta'' n(k+\nu(l))} \mu(n)\mu(l) V^{m+k+\nu(n)+\nu(l)} U^{n+l}. \end{aligned}$$

It follows that, for all $n, l, k \in \mathbb{Z}$,

$$\nu(n+l) = \nu(n) + \nu(l), \tag{143}$$
$$e^{ink(\theta'-\theta'')}\mu(n+l) = e^{i\theta'' n\nu(l)}\mu(n)\mu(l). \tag{144}$$

Condition (143) implies that, for all $n \in \mathbb{Z}$, $\nu(n) = \beta n$, where $\beta = \nu(1)$. Only left hand side of condition (144) depends on k, therefore it can be satisfied for all $k \in \mathbb{Z}$ only if $\theta' = \theta''$. We assume this, and then we have the following recursive relation

$$\mu(n+l) = e^{i\theta' n\beta l}\mu(n)\mu(l), \text{ for all } n, l \in \mathbb{Z}, \tag{145}$$

which has a family of solutions

$$\mu(n) = \alpha^n e^{i\beta\theta'\frac{n^2}{2}}, \text{ for all } n \in \mathbb{Z}, \tag{146}$$

where $\alpha \in \mathbb{C} \setminus \{0\}$. It follows that, for all $n \in \mathbb{Z}$, $\Gamma_1^2(u^n) = \alpha^n e^{i\beta\theta'\frac{n^2}{2}} V^{\beta n}$.

Similarly we prove that θ must equal θ'' and then, for all $n \in \mathbb{Z}$, $\Gamma_2^1(u^n) = (\alpha')^n e^{i\beta'\theta\frac{n^2}{2}} V^{\beta' n}$, for some $\alpha' \in \mathbb{C} \setminus \{0\}$ and $\beta' \in \mathbb{Z}$. In particular $\Gamma_2^1(u^n) = 1$, for all $n \in \mathbb{Z}$, is an admissible gauge transformation and, by the Remark after the Lemma 7.1, we can assume just that without losing any generality. Accordingly, the most general form of gluing maps for two quantum solid tori can be defined on the basis elements (cf. (133)) of respective solid tori as

$$\begin{aligned} \chi_2^1((1-xx^*)^k x^m h^n) &= \delta_{k0} V^m U^n, \\ \chi_1^2((1-yy^*)^k y^m g^n) &= \delta_{k0}\alpha^n e^{i\beta\theta\frac{n^2}{2}} V^{m+\beta n} U^n, \end{aligned} \tag{147}$$

for all $m, n \in \mathbb{Z}$, $k \in \mathbb{N}_0$. Note that χ_2^1 is a $*$-algebra map, and χ_1^2 is a $*$-algebra map if $|\alpha| = 1$. Observe that the glued algebra $P = \bigoplus_{\chi_j^i} P_i$ is a $*$-algebra in a natural way (i.e., with a $*$-operation defined by starring the components of the direct sum) if and only if the maps χ_j^i are $*$-algebra morphisms. On the other hand, scaling of g by a number of modulus one is an H-comodule $*$-algebra isomorphism of P_2. It follows that, if $|\alpha| = 1$, the parameter α can be absorbed, up to an isomorphism, by the redefinition $g \mapsto \alpha^{-1}g$. Accordingly, in what follows, we shall only consider the case $\alpha = 1$.

Let us denote the generators of the algebra $P_1^- = \vartheta(D_p \times_{-\theta} S^1)$ (resp. $P_2^- = \vartheta(D_q \times_{-\theta} S^1)$, $P_{12}^- = \vartheta(T_{-\theta})$) with the same symbols as the generators of P_1 (resp. P_2, P_{12}). We define, by the action on generators, the $*$-algebra isomorphisms

$$\begin{aligned} \eta_1 &: P_1 \to P_1^-, \;\; x \mapsto x,\; h \mapsto h^*, \\ \eta_2 &: P_2 \to P_2^-, \;\; y \mapsto y,\; g \mapsto g^*, \\ \eta_{12} &: P_{12} \to P_{12}^-, \;\; V \mapsto V,\; U \mapsto U^*. \end{aligned} \tag{148}$$

Clearly, for all $m, n \in \mathbb{Z}$, $k \in \mathbb{N}_0$,

$$\begin{aligned} \bar{\chi}_2^1 = \eta_{12} \circ \chi_2^1 \circ \eta_1^{-1} &: P_1^- \to P_{12}^-, \;\; (1-xx^*)^k x^m h^n \mapsto \delta_{k0} V^m U^n, \\ \bar{\chi}_1^2 = \eta_{12} \circ \chi_1^2 \circ \eta_2^{-1} &: P_2^- \to P_{12}^-, \;\; (1-xx^*)^k x^m h^n \mapsto \delta_{k0} e^{i\beta\theta\frac{n^2}{2}} V^{m-\beta n} U^n. \end{aligned} \tag{149}$$

Thus maps $\bar{\chi}_2^1$, $\bar{\chi}_1^2$ have the same form as maps (147) after substituting $\theta \mapsto -\theta$, $\beta \mapsto -\beta$. Denote $P^- = P_1^- \oplus_{\bar{\chi}_j^i} P_2^-$. It follows that the map

$$\eta = \eta_1 \oplus \eta_2 : P \to P^- \tag{150}$$

is a $*$-algebra isomorphism. Consequently, without losing generality, in what follows we shall confine ourselves to the case $\beta \geq 0$.

Using (147) and Lemma 6.7, it is easy to see, that the vectors

$$((1-xx^*)^k x^m h^n, 0), \ \ (0, (1-yy^*)^k y^m g^n), \ \ (x^m h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{m-\beta n} g^n), \tag{151}$$

$m, n \in \mathbb{Z}$, $k > 0$, form a basis of P.

Lemma 8.1. *The elements*

$$\xi = (1-xx^*, 0), \ z = (x, y), \ a = (e^{\frac{i\beta\theta}{2}} x^\beta h, g), \ b = (e^{\frac{i\beta\theta}{2}} h^{-1}, y^\beta g^{-1}) \tag{152}$$

of P generate P as a $*$-algebra.

Proof. It is enough to show that the basis vectors (151) are expressible in terms of elements (152). Observe that

$$(0, 1-yy^*) = 1 - zz^* - \xi. \tag{153}$$

It follows immediately that, for all $k > 0$, $m, n \in \mathbb{Z}$,

$$((1-xx^*)^k x^m h^n, 0) = e^{\frac{i\beta\theta}{2}n} \xi^k z^m b^{-n}, \tag{154}$$

and

$$(0, (1-yy^*)^k y^m g^n) = (1 - zz^* - \xi)^k z^m a^n. \tag{155}$$

Furthermore, using equation (125), for all $m, n \in \mathbb{Z}$,

$$\begin{aligned}
&(x^m h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{m-\beta n} g^n) \\
&\quad = (x^m h^n, e^{-i\beta\theta\frac{n^2}{2}} y^m y^{-\beta n} g^n) - (0, e^{-i\beta\theta\frac{n^2}{2}} y^{m-\beta n} Q^q_{m;-\beta n}(1-yy^*) g^n) \\
&\qquad\quad = z^m (h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{-\beta n} g^n) - e^{-i\beta\theta\frac{n^2}{2}} z^{m-\beta n} Q^q_{m;\beta n}(1 - zz^* - \xi) a^n.
\end{aligned}$$

As $y^{-\beta n} g^n = e^{i\beta\theta\frac{n(n-1)}{2}} (y^{-\beta} g)^n$, it follows that

$$(h^n, e^{-i\beta\theta\frac{n^2}{2}} y^{-\beta n} g^n)(h, e^{-\frac{i\beta\theta}{2}} y^{-\beta} g)^n = e^{\frac{i\beta\theta}{2}n} b^{-n}.$$

$\square$

Lemma 8.2. *The generators ξ, z, a, b of P satisfy the following relations.*

$$\xi^* = \xi, \;\; \xi z = pz\xi, \;\; z^*z - qzz^* = 1 - q - (p-q)\xi, \tag{156a}$$

$$(1 - zz^* - \xi)\xi = 0, \tag{156b}$$

$$\xi a = p^\beta a\xi, \;\; \xi b = b\xi, \;\; za = e^{-i\theta}az, \;\; zb = e^{i\theta}bz, \tag{156c}$$

$$za^* - e^{i\theta}a^*z = (p^\beta - 1)\xi z^{1-\beta}b, \tag{156d}$$

$$z^*b - e^{-i\theta}bz^* = (1-q^\beta)z^{\beta-1}(1 - zz^* - \xi)a^*, \tag{156e}$$

$$ab = e^{i\beta\theta}ba, \;\; ab^* = e^{-i\beta\theta}b^*a, \tag{156f}$$

$$ba = z^\beta, \tag{156g}$$

$$a^*a = \sum_{m=0}^{\beta}(-1)^m p^{\beta m - \frac{m(m-1)}{2}}\begin{bmatrix}\beta\\ m\end{bmatrix}_{p^{-1}}\xi^m, \tag{156h}$$

$$aa^* = \sum_{m=0}^{\beta}(-1)^m p^{-\beta m + \frac{m(m+1)}{2}}\begin{bmatrix}\beta\\ m\end{bmatrix}_{p}\xi^m, \tag{156i}$$

$$b^*b = \sum_{m=0}^{\beta}(-1)^m q^{\beta m - \frac{m(m-1)}{2}}\begin{bmatrix}\beta\\ m\end{bmatrix}_{q^{-1}}(1 - zz^* - \xi)^m, \tag{156j}$$

$$bb^* = \sum_{m=0}^{\beta}(-1)^m q^{-\beta m + \frac{m(m+1)}{2}}\begin{bmatrix}\beta\\ m\end{bmatrix}_{q}(1 - zz^* - \xi)^m. \tag{156k}$$

where $\begin{bmatrix}n\\ m\end{bmatrix}_p$ *are deformed binomial coefficients defined in (124)*

Proof. Easy if tedious proof is left to the reader. □

By the discussion in Section 4, the algebra P is naturally an H-comodule $*$-algebra. The coaction $\rho^H : P \to P \otimes H$ is defined on generators by

$$\rho^H(\xi) = \xi \otimes 1, \;\; \rho^H(z) = z \otimes 1, \;\; \rho^H(a) = a \otimes u, \;\; \rho^H(b) = b \otimes u^{-1}. \tag{157}$$

It is clear (cf. discussion around eq. (82)) that $P^{\mathrm{co}H} = B = B_1 \oplus_{\pi^i_j} B_2$. It follows that $P^{\mathrm{co}H}$ is generated by the elements $\xi, z \in P$.

8.5 Lens spaces of positive charge

Let $p, q \in (0,1)$, $\theta \in [0, 2\pi)$, $\beta \in \mathbb{N}_0$, and let $\vartheta(L^{p,q,\theta}_\beta)$ be the quotient of a free $*$-algebra generated by the elements ξ, z, a, b, modulo the relations (156). We will call $\vartheta(L^{p,q,\theta}_\beta)$ a *coordinate algebra of functions on a quantum lens space* $L^{p,q,\theta}_\beta$ *of positive charge* β.

Consider the family

$$\{\xi^k z^m b^n \mid k > 0, \; m, n \in \mathbb{Z}\}, \;\; \{(1 - zz^* - \xi)^k z^m a^n \mid k \geq 0, \; m, n \in \mathbb{Z}\} \tag{158}$$

of vectors in $\vartheta(L_\beta^{p,q,\theta})$. We will prove that it is a basis of $\vartheta(L_\beta^{p,q,\theta})$. First we need to prove several technical lemmas. Let

$$\mathcal{A} = (\text{ Span of the family (158) }). \tag{159}$$

Lemma 8.3. *The elements $1_\mathcal{A}$, ξ, z, z^*, a, a^*, b, b^* belong to $\mathcal{A}$.*

Proof. The assertion is obvious in the case of the elements $1_\mathcal{A}$, ξ, z, z^*, a, a^*. Furthermore,

$$\begin{aligned} b = b\left(aa^* - \sum_{m=1}^{\beta}(-1)^m p^{-\beta m+\frac{m(m+1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_p \xi^m\right) \\ = z^\beta a^* - \sum_{m=1}^{\beta}(-1)^m p^{-\beta m+\frac{m(m+1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_p \xi^m b, \end{aligned}$$

where we used eq. (156i) in the first equality, and eq. (156g) in the second. Therefore $b \in \mathcal{A}$. Similarly, using eq. (156h) and eq. (156g), we obtain

$$b^* = e^{-i\beta\theta} z^{-\beta} a - \sum_{m=1}^{\beta}(-1)^m p^{\beta m-\frac{m(m-1)}{2}} \begin{bmatrix} \beta \\ m \end{bmatrix}_{p^{-1}} \xi^m b^* \in \mathcal{A}.$$

□

Lemma 8.4. *The following relations are satisfied in $\vartheta(L_\beta^{p,q,\theta})$:*

$$(1 - zz^* - \xi)z = qz(1 - zz^* - \xi), \tag{160a}$$
$$(1 - zz^* - \xi)a = a(1 - zz^* - \xi), \tag{160b}$$
$$(1 - zz^* - \xi)b = q^\beta b(1 - zz^* - \xi), \tag{160c}$$
$$\xi a^n = e^{i\beta\theta\frac{n(n+1)}{2}} \xi z^{\beta n} b^{-n}, \text{ for all } n \in \mathbb{Z}. \tag{160d}$$

Proof. Using (156a) yields

$$\begin{aligned}(1 - zz^* - \xi)z &= z - z(z^*z) - pz\xi \\ &= z - z(1 - q(1 - zz^* - \xi) - p\xi) - pz\xi = qz(1 - zz^* - \xi).\end{aligned}$$

Furthermore,

$$\begin{aligned}(1 - zz^* - \xi)a &= a - z(z^*a) - p^\beta a\xi = a - e^{i\theta}z(az^* + (1 - p^\beta)b^*z^{\beta-1}\xi) - p^\beta a\xi \\ &= a - azz^* - e^{i\beta\theta}(1 - p^\beta)z^\beta b^*\xi - p^\beta a\xi \\ &= a - azz^* - (1 - p^\beta)abb^*\xi - p^\beta a\xi = a(1 - zz^* - \xi),\end{aligned}$$

where in the forth equality we used eq. (156g). Similar proof of the equation (160c) is left to the reader as an exercise.

To prove the property (160d), we note that, by (160c), (156j) and (156k),

$$\xi b^n b^{-n} = 1, \text{ for all } n \in \mathbb{Z}. \tag{161}$$

Therefore, for all $n \in \mathbb{Z}$,

$$e^{i\beta\theta\frac{n(n+1)}{2}}\xi z^{\beta n}b^{-n} = e^{i\beta\theta\frac{n(n+1)}{2}}\xi(ba)^n b^{-n} = \xi a^n b^n b^{-n} = \xi a^n.$$

□

Lemma 8.5. *The vector subspace $\mathcal{A} \subseteq \vartheta(L_\beta^{p,q,\theta})$ (eq. (159)) is closed under multiplication.*

Proof. It is enough to consider products of basis vectors (158). Observe that, by equations (160a)–(160c),

$$\{\xi^k z^m b^n \mid k \in \mathbb{N},\ m,n \in \mathbb{Z}\} \cdot \{(1 - zz^* - \xi)^l z^s a^t \mid l \in \mathbb{N},\ s,t \in \mathbb{Z}\} = \{0\},$$

and

$$\{(1 - zz^* - \xi)^l z^s a^t \mid l \in \mathbb{N},\ s,t \in \mathbb{Z}\} \cdot \{\xi^k z^m b^n \mid k \in \mathbb{N},\ m,n \in \mathbb{Z}\} = \{0\}.$$

Note that $zz^* = 1 - (1 - zz^* - \xi) - \xi$, and, by (156a),

$$z^* z = 1 - q(1 - zz^* - \xi) - p\xi.$$

It follows, using (156a), (156b) and (160a), that, for all $n, m \in \mathbb{Z}$,

$$z^n z^m = (1 + \mathcal{P}_{n,m}(\xi) + \mathcal{Q}_{n,m}(1 - zz^* - \xi))z^{n+m}, \tag{162a}$$

where $\mathcal{P}_{n,m}$ and $\mathcal{Q}_{n,m}$ are polynomials such that $\mathcal{P}_{n,m}(0) = \mathcal{Q}_{n,m}(0) = 0$. Similarly by relations (156h)–(156k), (156c) and (160b)–(160c), for all $n, m \in \mathbb{Z}$,

$$a^n a^m = (1 + \mathcal{P}'_{n,m}(\xi))a^{n+m}, \tag{162b}$$

$$b^n b^m = (1 + \mathcal{Q}'_{n,m}(1 - zz^* - \xi))b^{n+m}, \tag{162c}$$

where polynomials $\mathcal{P}'_{n,m}$, $\mathcal{Q}'_{n,m}$, satisfy $\mathcal{P}'_{n,m}(0) = \mathcal{Q}'_{n,m}(0) = 0$.

It follows that, for all $m, n, s, t \in \mathbb{Z}$ and $k, l \in \mathbb{N}$,

$$\begin{aligned}(\xi^k z^m b^n)(\xi^l z^s b^t) &= p^{-lm} e^{-ins\theta} \xi^{k+l} z^m z^s b^{n+t} \\ &= p^{-lm} e^{-ins\theta} \xi^{k+l}(1 + \mathcal{P}_{m,s}(\xi))z^{m+s} b^{n+t} \\ &\in \mathrm{Span}(\{\xi^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\}) \subseteq \mathcal{A}.\end{aligned} \tag{163}$$

Using eq. (160d) yields, for all $m, n, s, t \in \mathbb{Z}$, $k \in \mathbb{N}$,

$$(\xi^k z^m b^n)(z^s a^t) = e^{i\beta\theta\frac{t(t+1)}{2}} \xi^k z^m b^n z^s z^{\beta t} b^{-t},$$

hence, by eq. (163),

$$(\xi^k z^m b^n)(z^s a^t) \in \mathrm{Span}(\{\xi^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\}) \subseteq \mathcal{A}. \tag{164}$$

Analogously,

$$\begin{aligned}(z^s a^t)(\xi^k z^m b^n) &= p^{-\beta t k} z^s \xi^k a^t z^m b^n = e^{i\beta\theta\frac{t(t+1)}{2}} p^{-\beta t k} z^s \xi^k z^{\beta t} b^{-t} z^m b^n \\ &\in \mathrm{Span}(\{\xi^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\}) \subseteq \mathcal{A}.\end{aligned} \tag{165}$$

In the remainder of the proof we need the following observation. For all $m, n \in \mathbb{Z}$,

$$a^n z^m \in e^{imn\theta} z^m a^n + \mathrm{Span}(\{\xi^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\}). \tag{166}$$

We use induction on $m, n \in \mathbb{Z}$. The above formula is obviously true for m or n equal to zero. By eq. (156c) and eq. (156d), it is also true for $m, n = \pm 1$. For brevity write $\mathcal{A}' = \mathrm{Span}(\{\xi^k z^m b^n \mid k \in \mathbb{N}, m, n \in \mathbb{Z}\})$. Let $k = \pm 1$, $kn \geq 0$. Then, using equations (163), (164), (165), we obtain

$$\begin{aligned} a^{n+k} z^m = a^k (a^n z^m) \in e^{imn\theta} (a^k z^m) a^n + a^k \mathcal{A}' \subseteq e^{imn\theta} (a^k z^m) a^n + \mathcal{A}' \\ \subseteq e^{im(n+k)\theta} z^m a^{n+k} + e^{imn\theta} \mathcal{A}' a^n + \mathcal{A}' \subseteq e^{im(n+k)\theta} z^m a^{n+k} + \mathcal{A}'. \end{aligned}$$

Similarly, for $k = \pm 1$, $km \geq 0$,

$$\begin{aligned} a^n z^{m+k} = (a^n z^m) z^k \in e^{imn\theta} z^m (a^n z^k) + \mathcal{A}' z^k \subseteq e^{imn\theta} z^m (e^{ink\theta} z^k a^n + \mathcal{A}') + \mathcal{A}' z^k \\ \subseteq e^{i(m+k)n\theta} z^{m+k} a^n + \mathcal{A}'. \end{aligned}$$

Using (166) and then (162) and (160), we obtain, for all $m, n, s, t \in \mathbb{Z}$,

$$\begin{aligned} z^m a^n z^s a^t &\in e^{ins\theta} z^m z^s a^n a^t + z^m \mathcal{A}' a^t \\ &\subseteq e^{ins\theta} (1 + \mathcal{Q}_{m,s}(1 - zz^* - \xi) + \mathcal{P}_{m,s}(\xi)) z^{m+s} (1 + \mathcal{P}'_{n,t}(\xi)) a^{n+t} + \mathcal{A}' \\ &= e^{ins\theta} (1 + \mathcal{Q}_{m,s}(1 - zz^* - \xi)) z^{m+s} a^{n+t} + \mathcal{P}''(\xi) z^{m+s+\beta(n+t)} b^{-(n+t)} + \mathcal{A}' \subseteq \mathcal{A}, \end{aligned}$$

where $\mathcal{P}''$ is a polynomial such that $\mathcal{P}''(0) = 0$, and in the last equality we used eq. (160d) and then eq. (162a). This shows that, for all $m, n, s, t \in \mathbb{Z}$ and $k, l \in \mathbb{N}_0$, $((1 - zz^* - \xi)^k z^m a^n)((1 - zz^* - \xi)^l z^s a^t) \in \mathcal{A}$, which ends the proof. □

Proposition 8.6. *Vectors (158) form a basis of $\vartheta(L^{p,q,\theta}_\beta)$. The algebras $\vartheta(L^{p,q,\theta}_\beta)$ and $P = P_1 \oplus_{\chi^i_j} P_2 = \vartheta(D_p \times_\theta S^1) \oplus_{\chi^i_j} \vartheta(D_q \times_\theta S^1)$ are mutually isomorphic. Here the maps χ^1_2 and χ^2_1 are defined in eq. (147) with $\alpha = 1$ and $\beta \geq 0$.*

Proof. Let the algebra maps $\chi_i : \vartheta(L^{p,q,\theta}_\beta) \to P_i$, $i = 1, 2$, be defined on generators by

$$\begin{aligned} \chi_1(\xi) = 1 - xx^*, \ \ \chi_1(z) = x, \ \ \chi_1(a) = e^{\frac{i\beta\theta}{2}} x^\beta h, \ \ \chi_1(b) = e^{\frac{i\beta\theta}{2}} h^{-1}, \\ \chi_2(\xi) = 0, \ \ \chi_2(z) = y, \ \ \chi_2(a) = g, \ \ \chi_2(b) = y^\beta g^{-1}. \end{aligned} \tag{167}$$

By Lemmas 8.1 and 8.2 these maps are well defined, and by Lemma 8.1, the map $\chi = \chi_1 \oplus \chi_2 : \vartheta(L^{p,q,\theta}_\beta) \to P$ is surjective. Let $w \in \ker \chi$. By Lemmas 8.3 and 8.5, vectors (158) span $\vartheta(L^{p,q,\theta}_\beta)$, hence

$$w = \sum_{\substack{m,n \in \mathbb{Z} \\ k > 0}} \mu_{kmn} \xi^k z^m b^n + \sum_{\substack{s,t \in \mathbb{Z} \\ l \geq 0}} \nu_{lst} (1 - zz^* - \xi)^l z^s a^t, \tag{168}$$

for some coefficients $\mu_{kmn}, \nu_{lst} \in \mathbb{C}$, where $m, n, s, t \in \mathbb{Z}$, $k > 0$, $l \geq 0$. By assumption, $\chi_1(w) = 0$ and $\chi_2(w) = 0$. It follows that

$$0 = \chi_2(w) = \sum_{\substack{s,t\in\mathbb{Z}\\ l\geq 0}} \nu_{lst}(1 - yy^*)^l y^s g^t.$$

Since the elements $(1 - yy^*)^l y^s g^t$, $l \in \mathbb{N}_0$, $s, t \in \mathbb{Z}$, form a linear basis of P_2, this implies that, for all $l \in \mathbb{N}_0$, $s, t \in \mathbb{Z}$, $\nu_{lst} = 0$. But then

$$0 = \chi_1(w) = \sum_{\substack{m,n\in\mathbb{Z}\\ k>0}} \mu_{kmn} e^{\frac{i\beta\theta}{2}n}(1 - xx^*)^k x^m h^n,$$

which implies that, for all $k \in \mathbb{N}$ and $m, n \in \mathbb{Z}$, $\mu_{kmn} e^{\frac{i\beta\theta}{2}n} = 0$ and so $\mu_{kmn} = 0$. Hence $w = 0$ and therefore $\ker \chi = \{0\}$ and so $\chi : \vartheta(L_\beta^{p,q,\theta}) \to P$ is a $*$-algebra isomorphism. It follows that we can identify $\vartheta(L_\beta^{p,q,\theta})$ with P. We have also proven that vectors (158) are linearly independent and hence they form a linear basis of $\vartheta(L_\beta^{p,q,\theta})$.

Let us define a right H-coaction $\rho^H : \vartheta(L_\beta^{p,q,\theta}) \to \vartheta(L_\beta^{p,q,\theta}) \otimes H$ by (eq. 157), which makes $\chi : \vartheta(L_\beta^{p,q,\theta}) \to P$ an H-comodule isomorphism. It follows, by the discussion after eq. (157), that $B = P^{\mathrm{co}H}$ is isomorphic to the quotient of a free algebra, generated by elements, ξ, z, by the relations (156a)-(156b). This, in turn, is the coordinate algebra $\vartheta(S^2_{pq})$ on the quantum 2-sphere S^2_{pq} defined, by gluing two quantum discs D_p and D_q, in [8] and [13]. □

8.6 The inverse of the canonical map on $\vartheta(L_\beta^{p,q,\theta})$

By Proposition 6.9, $P(B)^H$ is an H-Hopf Galois extension. The translation maps on P_1 and P_2 are given explicitly, by the formulae, for all $n \in \mathbb{Z}$,

$$\begin{aligned} \tau_1 &: H \to P_1 \otimes_B P_1, \ \ u^n \mapsto h^{-n} \otimes_B h^n, \\ \tau_2 &: H \to P_2 \otimes_B P_2, \ \ u^n \mapsto g^{-n} \otimes_B g^n. \end{aligned} \tag{169}$$

By eq. (41), the translation map on P is given explictly as, for all $n \in \mathbb{Z}$,

$$\tau : H \to P \otimes_B P, \ \ u^n \mapsto \kappa^{-1}_{P\otimes_B P}(\tau_1(u^n), \tau_2(u^n)), \tag{170}$$

i.e., for all $n \in \mathbb{Z}$, the element $\tau(u^n) \in P \otimes_B P$ is uniquely determined by the property

$$(\chi_1 \otimes_B \chi_1)(\tau(u^n)) = h^{-n} \otimes_B h^n, \ \ (\chi_2 \otimes_B \chi_2)(\tau(u^n)) = g^{-n} \otimes_B g^n, \tag{171}$$

where maps χ_1, χ_2 were defined in (167). In order to find $\tau(u^n)$, first note that, for all $n \in \mathbb{Z}$,

$$(\chi_1 \otimes_B \chi_1)(b^n \otimes_B b^{-n}) = h^{-n} \otimes_B h^n, \ \ (\chi_2 \otimes_B \chi_2)(a^{-n} \otimes_B a^n) = g^{-n} \otimes_B g^n. \tag{172}$$

Then, for all $n \in \mathbb{Z}$,

$$\begin{aligned} b^n \otimes_B b^{-n} &= a^{-n} a^n b^n \otimes_B b^{-n} + (1 - a^{-n} a^n) b^n \otimes_B b^{-n} \\ &= a^{-n} \otimes_B a^n b^n b^{-n} + (1 - a^{-n} a^n) b^n \otimes_B b^{-n} \\ &= a^{-n} \otimes_B a^n + a^{-n} \otimes_B a^n (b^n b^{-n} - 1) + (1 - a^{-n} a^n) b^n \otimes_B b^{-n}. \end{aligned} \tag{173}$$

Observe that, for all $n \in \mathbb{Z}$, $1 - a^{-n} a^n \in \ker \chi_2$ and $b^n b^{-n} - 1 \in \ker \chi_1$, therefore the elements

$$\begin{aligned} \tau(u^n) &= b^n \otimes_B b^{-n} + a^{-n} \otimes_B a^n (1 - b^n b^{-n}) \\ &= a^{-n} \otimes_B a^n + (1 - a^{-n} a^n) b^n \otimes_B b^{-n}, \end{aligned} \tag{174}$$

$n \in \mathbb{Z}$, satisfy conditions (171), and hence define the translation map on P.

The above method of computation was inspired by the proof of Proposition 1 in [8].

8.7 Representations of $\vartheta(L_\beta^{p,q,\theta})$

To find representations of $\vartheta(L_\beta^{p,q,\theta})$ we use the same method as was used in [12]. Let $\varrho : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H})$ be any representation of $\vartheta(L_\beta^{p,q,\theta})$ as a subalgebra of the algebra of bounded operators $\mathbf{B}(\mathcal{H})$ on a Hilbert space $\mathcal{H}$. Note that, by the relations (160), (156a) and (156c), the subspaces $\ker \varrho(\xi)$ and $\ker \varrho(1 - zz^* - \xi)$ are invariant. For any pair of closed subspaces $\mathcal{S} \subseteq \mathcal{S}' \subseteq \mathcal{H}$, let $\mathcal{S}^{\perp_{\mathcal{S}'}}$ denote the closure of the orthogonal complement of $\mathcal{S}$ in $\mathcal{S}'$. For brevity, we denote $\mathcal{S}^\perp = \mathcal{S}^{\perp_{\mathcal{H}}}$. In this section symbol '$\oplus$' denotes the orthogonal direct sum of Hilbert spaces. Hilbert space $\mathcal{H}$ can be decomposed into a direct sum

$$\begin{aligned} \mathcal{H} = \ker \varrho(\xi) \oplus (\ker \varrho(\xi))^\perp &= (\ker \varrho(\xi) \cap \ker \varrho(1 - zz^* - \xi)) \\ &\oplus (\ker \varrho(\xi) \cap \ker \varrho(1 - zz^* - \xi))^{\perp_{\ker \varrho(\xi)}} \oplus (\ker \varrho(\xi))^\perp. \end{aligned}$$

Suppose that $\Psi \in (\ker \varrho(\xi))^\perp$ is such that $\varrho(1 - zz^* - \xi)\Psi \neq 0$. Since $(\ker \varrho(\xi))^\perp$ is an invariant subspace, we obtain, by the relation (156b),

$$0 = \varrho(\xi(1 - zz^* - \xi))\Psi = \varrho(\xi)\varrho(1 - zz^* - \xi)\Psi \neq 0,$$

which is a contradiction. It follows that $(\ker \varrho(\xi))^\perp \subseteq \ker \varrho(1 - zz^* - \xi)$. For brevity, let us denote $\mathcal{H}_0 = \ker \varrho(\xi) \cap \ker \varrho(1 - zz^* - \xi)$, $\mathcal{H}' = (\ker \varrho(\xi) \cap \ker \varrho(1 - zz^* - \xi))^{\perp_{\ker \varrho(\xi)}}$, $\mathcal{H}'' = (\ker \varrho(\xi))^\perp$. It follows that $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}' \oplus \mathcal{H}''$, and we have an orthogonal direct sum decomposition of the representation ϱ into subrepresentations

$$\varrho_0 : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}_0), \;\; \varrho' : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}'), \;\; \varrho'' : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}'').$$

In the representation ϱ_0, the relations (156) are reduced to

$$\varrho_0(\xi) = 0, \;\; \varrho_0(b) = \varrho_0(z)^\beta \varrho_0(a^*), \tag{175a}$$

$$\varrho_0(z^*)\varrho_0(z) = 1 = \varrho_0(z)\varrho_0(z^*), \;\; \varrho_0(a^*)\varrho_0(a) = 1 = \varrho_0(a)\varrho_0(a^*), \tag{175b}$$

$$\varrho_0(a)\varrho_0(z^{\pm 1}) = e^{\pm i\theta}\varrho_0(z^{\pm 1})\varrho_0(a). \tag{175c}$$

Similarly, in the representation ϱ', the relations (156) assume the form

$$\varrho'(\xi) = 0, \ \ \varrho'(b) = \varrho'(z)^{\beta}\varrho'(a^*), \tag{176a}$$
$$\varrho'(z^*)\varrho'(z) - q\varrho'(z)\varrho'(z^*) = 1 - q, \ \ \varrho'(a^*)\varrho'(a) = 1 = \varrho'(a)\varrho'(a^*), \tag{176b}$$
$$\varrho'(a)\varrho'(z^{\pm 1}) = e^{\pm i\theta}\varrho'(z^{\pm 1})\varrho'(a). \tag{176c}$$

Finally, the representation ϱ'' reduces the relations (156) to the form

$$\varrho''(\xi) = 1 - \varrho''(z)\varrho''(z^*), \ \ \varrho''(a) = \varrho''(b^*)\varrho''(z)^{\beta}, \tag{177a}$$
$$\varrho''(z^*)\varrho''(z) - p\varrho''(z)\varrho''(z^*) = 1 - p, \ \ \varrho''(b^*)\varrho''(b) = 1 = \varrho''(b)\varrho''(b^*), \tag{177b}$$
$$\varrho''(z)\varrho''(b^{\pm 1}) = e^{\pm i\theta}\varrho''(b^{\pm 1})\varrho''(z). \tag{177c}$$

¿From the representation theory of the quantum solid torus (Section 8.3), it follows that irreducible representations of $\vartheta(L_{\beta}^{p,q,\theta})$ include the ones unitarily equivalent to one of the following representations.

For all $0 \leq \mu < 2\pi$, there exists a representation $\varrho'_{\mu} : \vartheta(L_{\beta}^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}'_{\mu})$, where $\mathcal{H}'_{\mu}$ has an orthonormal Hilbert basis Ψ'_n, $n \in \mathbb{N}_0$, such that, for all $n \in \mathbb{N}_0$,

$$\begin{gathered}
\varrho'_{\mu}(z)\Psi'_n = \sqrt{1-q^{n+1}}\Psi'_{n+1}, \ \ \varrho'_{\mu}(z^*)\Psi'_n = \sqrt{1-q^{n}}\Psi'_{n-1} \text{ if } n > 0, \ \ \varrho'_{\mu}(z^*)\Psi'_0 = 0, \\
\varrho'_{\mu}(a^{\pm 1})\Psi'_n = e^{\pm i(\mu + n\theta)}\Psi'_n, \ \ \varrho'_{\mu}(\xi)\Psi'_n = 0, \\
\varrho'_{\mu}(b)\Psi'_n = e^{-i(\mu+n\theta)}\sqrt{1-q^{n+1}}\ldots\sqrt{1-q^{n+\beta}}\Psi'_{n+\beta}, \\
\varrho'_{\mu}(b^*)\Psi'_n \begin{cases} 0 & \text{if } n < \beta, \\ e^{i(\mu+(n-\beta)\theta)}\sqrt{1-q^{n}}\ldots\sqrt{1-q^{n-\beta+1}}\Psi'_{n-\beta} & \text{otherwise.} \end{cases}
\end{gathered} \tag{178}$$

Similarly, for all $0 \leq \mu < 2\pi$, there exists a representation $\varrho''_{\mu} : \vartheta(L_{\beta}^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}''_{\mu})$, where $\mathcal{H}''_{\mu}$ has an orthonormal Hilbert basis Ψ'_n, $n \in \mathbb{N}_0$, such that, for all $n \in \mathbb{N}_0$,

$$\begin{gathered}
\varrho''_{\mu}(z)\Psi''_n = \sqrt{1-p^{n+1}}\Psi''_{n+1}, \ \ \varrho''_{\mu}(z^*)\Psi''_n = \sqrt{1-p^{n}}\Psi''_{n-1} \text{ if } n > 0, \ \ \varrho''_{\mu}(z^*)\Psi''_0 = 0, \\
\varrho''_{\mu}(b^{\pm 1})\Psi''_n = e^{\pm i(\mu - n\theta)}\Psi''_n, \ \ \varrho''_{\mu}(\xi)\Psi''_n = p^n\Psi''_n, \\
\varrho''_{\mu}(a)\Psi''_n = e^{-i(\mu-(n+\beta)\theta)}\sqrt{1-p^{n+1}}\ldots\sqrt{1-p^{n+\beta}}\Psi''_{n+\beta}, \\
\varrho''_{\mu}(a^*)\Psi''_n \begin{cases} 0 & \text{if } n < \beta, \\ e^{i(\mu-n\theta)}\sqrt{1-p^{n}}\ldots\sqrt{1-p^{n-\beta+1}}\Psi''_{n-\beta} & \text{otherwise.} \end{cases}
\end{gathered} \tag{179}$$

Finally, depending on whether deformation parameter θ is a rational or irrational multiple of 2π, we have one of the following families of irreducible representations.

Suppose that $\theta = 2\pi\frac{M}{N}$, where $M, N \in \mathbb{Z}$, $N > 0$, and M and N are relatively prime. Then, for all $0 \leq \mu, \nu < 2\pi$, there exists a representation $\varrho_0^{\mu\nu} : \vartheta(L_{\beta}^{p,q,\theta}) \to$

$\mathbf{B}(\mathcal{H}_0^{\mu\nu})$, where $\mathcal{H}_0^{\mu\nu}$ has an orthonormal Hilbert basis Ψ_n, $n \in \mathbb{Z}_N$, and, for all $n \in \mathbb{Z}_N$,

$$\begin{aligned} \varrho_0^{\mu\nu}(a^{\pm 1})\Psi_n e^{\pm i\frac{\mu}{N}\pm in\theta}\Psi_n, \;\; \varrho_0^{\mu\nu}(z^{\pm 1})\Psi_n = e^{\pm i\frac{\nu}{N}}\Psi_{n\pm 1}, \;\; \varrho_0^{\mu\nu}(\xi)\Psi_n = 0, \\ \varrho_0^{\mu\nu}(b)\Psi_n e^{i\frac{\nu\beta-\mu}{N}-in\theta}\Psi_{n+\beta}, \;\; \varrho_0^{\mu\nu}(b^*)\Psi_n e^{i\frac{\mu-\nu\beta}{N}+i(n-\beta)\theta}\Psi_{n-\beta}. \end{aligned} \tag{180}$$

If θ is an irrational multiple of 2π, then, for any $0 \leq \mu < 2\pi$, we have a representation $\varrho_0^{\mu} : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}_0^{\mu})$. The Hilbert space $\mathcal{H}_0^{\mu}$ has an orthonormal basis Ψ_n, $n \in \mathbb{Z}$. For all $n \in \mathbb{Z}$,

$$\begin{aligned} \varrho_0^{\mu}(a^{\pm 1})\Psi_n = e^{\pm i\mu\pm in\theta}\Psi_n, \;\; \varrho_0^{\mu}(z^{\pm 1})\Psi_n = \Psi_{n\pm 1}, \;\; \varrho_0^{\mu}(\xi)\Psi_n = 0, \\ \varrho_0^{\mu}(b)\Psi_n = e^{-i\mu-in\theta}\Psi_{n+\beta}, \;\; \varrho_0^{\mu}(b^*)\Psi_n = e^{i\mu+i(n-\beta)\theta}\Psi_{n-\beta}. \end{aligned} \tag{181}$$

If θ is irrational, then for any $0 \leq \mu, \nu < 2\pi$, the representation $\varrho'_\mu \oplus \varrho''_\nu : \vartheta(L_\beta^{p,q,\theta}) \to \mathbf{B}(\mathcal{H}'_\mu \oplus \mathcal{H}''_\nu)$ is faithful.

8.8 Final remarks

We conclude this section with a number of remarks about the structure of quantum lens spaces. We also comment on the K-theory of quantum lens spaces. The detailed developments of the topics discussed here, can be considered as directions for future work.

The relations (156) defining $\vartheta(L_\beta^{p,q,\theta})$ assume a particularly simple form in the case $\beta = 1$. Namely, if $\beta = 1$, then $z = ba$, $\xi = 1 - aa^*$, and the relations (156) are reduced to

$$\begin{aligned} a^*a - paa^* = 1 - p, \quad b^*b - qbb^* = 1 - q, \quad ab = e^{i\theta}ba, \quad ab^* = e^{-i\theta}b^*a, \\ (1 - aa^*)(1 - bb^*) = 0. \end{aligned} \tag{182}$$

It follows that the quantum lens space of charge 1, $L_1^{p,q,\theta}$, can be identified with the Heegaard quantum sphere $S^3_{p,q,\theta}$ considered in [2].

For brevity, let us denote $A = \vartheta(S^3_{p,q,\theta})$. Let us define a $\mathbb{Z}$-grading on A with

$$\deg(a) = 1, \quad \deg(a^*) = -1, \quad \deg(b) = -1, \quad \deg(b^*) = 1, \tag{183}$$

and, for all $n \in \mathbb{Z}$, let $A_n = \{w \in A \mid \deg(w) = n\}$. For any $\beta \in \mathbb{N}$, define the subalgebra $A(\beta) = \bigoplus_{n\in\mathbb{Z}} A_{\beta n}$. It can be shown that the $*$-algebra map $f_\beta : \vartheta(L_\beta^{p,q,\beta\theta}) \to A(\beta)$, given on generators by

$$f_\beta(\xi) = 1 - aa^*, \quad f_\beta(z) = ba, \quad f_\beta(a) = a^\beta, \quad f_\beta(b) = e^{i\theta\frac{\beta(\beta-1)}{2}}b^\beta, \tag{184}$$

is a well defined $*$-algebra isomorphism.

In [2], it was demonstrated that $C(S^3_{p,q,\theta})$ is isomorphic as a C^*-algebra to a fibre product of two C^*-algebras isomorphic to quantum solid tori, and then the Mayer-Vietoris sequence was used to compute the K-theory of $C(S^3_{p,q,\theta})$ using the K-theory

of quantum solid tori. We expect that this method can be adapted to compute the K-theory of $C(L_{\beta}^{p,q,\theta})$. This is a direction for future work.

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